Resolution a la Kronheimer of C^3/ Gamma singularities and the Monge-Ampere equation for Ricci-flat Kaehler metrics in view of D3-brane solutions of supergravity

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RESOLUTION À LA KRONHEIMER OF $\mathbb{C}^3/\Gamma$ SINGULARITIES AND THE MONGE-AMPÈRE EQUATION FOR RICCI-FLAT KÄHLER METRICS IN VIEW OF D3-BRANE SOLUTIONS OF SUPERGRAVITY

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In affectionate and respectful memory of
Boris A. Dubrovin, a master of mathematics and life

Abstract

In this paper we analyze the relevance of the generalized Kronheimer construction for the gauge-gravity correspondence. We begin with the general structure of D3-brane solutions of type IIB supergravity on smooth manifolds Y^Γ that are supposed to be the crepant resolution of quotient singularities $\mathbb{C}^3/\Gamma$ with Γ a finite subgroup of $SU(3)$. We emphasize that non trivial 3-form fluxes require the existence of imaginary self-dual harmonic forms $\omega^{2,1}$. Although excluded in the classical Kronheimer construction they may be reintroduced by means of mass deformations. Next we concentrate on the other essential item for the D3-brane construction, namely, the existence of a Ricci-flat metric on Y^Γ. We study the issue of Ricci-flat Kähler metrics on such resolutions Y^Γ, with particular attention to the case $\Gamma = \mathbb{Z}_4$. We advance the conjecture that on the exceptional divisor of Y^Γ the Kronheimer Kähler metric and the Ricci-flat one, that is locally flat at infinity, coincide. The conjecture is shown to be true in the case of the Ricci-flat metric on $\mathrm{tot}K_{\mathbb{W}P[112]}$ that we construct, *i.e.* the total space of the canonical bundle of the weighted projective space $\mathbb{W}P[112]$, which is a partial resolution of $\mathbb{C}^3/\mathbb{Z}_4$. For the full resolution we have $Y^{\mathbb{Z}_4} = \mathrm{tot}\,K_{\mathbb{F}_2}$, where $\mathbb{F}_2$ is the second Hizebruch surface. We try to extend the proof of the conjecture to this case using the one-parameter Kähler metric on $\mathbb{F}_2$ produced by the Kronheimer construction as initial datum in a Monge-Ampère (MA) equation. We exhibit three formulations of this MA equation, one in terms of the Kähler potential, the other two in terms of the symplectic potential but with two different choices of the variables. In both cases one can establish a series solution in powers of the variable along the fibers of the canonical bundle. The main property of the MA equation is that it does not impose any condition on the initial geometry of the exceptional divisor, rather it uniquely determines all the subsequent terms as local functionals of this initial datum. Although a formal proof is still missing, numerical and analytical results support the conjecture. As a by-product of our investigation we have identified some new properties of this type of MA equations that we believe to be so far unknown.

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1 Introduction

We report on the advances obtained on the following special aspect of the gauge/gravity correspondence, within the context of quiver gauge–theories [1, 2, 3, 4, 5]: *the relevance of the generalized Kronheimer construction*[6, 7] for the resolution of $\mathbb{C}^3/\Gamma$ singularities. In particular, after an introduction about D3-brane supergravity solutions, we consider, within this framework, the issues of the construction of a Ricci-flat metric on the smooth resolution Y^Γ of $\mathbb{C}^3/\Gamma$. We begin with the general problem of establishing holographic dual pairs whose members are

A) a gauge theory living on a D3-brane world volume,

B) a classical D3-brane solution of type IIB supergravity in D=10 supergravity.

Gauge theories based on quiver diagrams have been extensively studied in the literature [1, 2, 3, 4, 5] in connection with the problem of establishing holographic dual pairs as described above. Indeed the quiver diagram is a powerful tool which encodes the data of a Kähler quotient describing the geometry of the six directions transverse to the brane. The linear data of such a Kähler (or HyperKähler) quotient are the menu of the dual supersymmetric gauge theory, as they specify:

1. the gauge group factors,
2. the matter multiplets,
3. the representation assignments of the latter with respect to the gauge group factors.

The possibility of testing the holographic principle [8, 9, 10, 11, 12] and resorting to the supergravity side of the correspondence in order to perform, *classically and in the bulk*, quantum calculations that pertain to the boundary gauge theory is tightly connected with the quiver approach whenever the classical brane solution has a conformal point corresponding to a limiting geometry of the following type:

$$M_D \;=\; \mathrm{AdS}_{p+2} \times \mathrm{SE}^{D-p-2} \tag{1.1}$$

In the above equation by AdS_{p+2} we have denoted anti de Sitter space in $p+2$-dimensions while SE^{D-p-2} stands for a Sasaki-Einstein manifold in $D-p-2$-dimensions [13].

Within the general scope of quivers a special subclass is that of McKay quivers that are group theoretically defined by the embedding of a finite discrete group Γ in an n-dimensional complex unitary group

$$\Gamma \hookrightarrow \mathrm{SU}(n) \tag{1.2}$$

and are associated with the resolution of $\mathbb{C}^n/\Gamma$ quotient singularities by means of a Kronheimer-like construction [15, 16, 17].

The case $n = 2$ corresponds to the HyperKähler quotient construction of ALE-manifolds as the resolution of the $\mathbb{C}^2/\Gamma$ singularities, the discrete group Γ being a finite Kleinian subgroup of SU(2), as given by the ADE classification[1].

The case $n = 3$ was the target of many interesting and robust mathematical developments starting from the middle of the nineties up to the present day [19, 20, 21, 22, 23, 24, 25, 26, 27]. The main and most intriguing result in this context, which corresponds to a generalization of the Kronheimer construction and of the McKay correspondence, is the group theoretical prediction of the cohomology groups $\mathrm{H}^{(p,q)}\left(Y^\Gamma_{[3]}\right)$ of the crepant smooth resolution $Y^\Gamma_{[3]}$ of the quotient singularity $\mathbb{C}^3/\Gamma$. Specifically, the main output of the generalized Kronheimer construction for the crepant resolution of a singularity $\mathbb{C}^3/\Gamma$ is a blowdown morphism:

$$\mathrm{BD}\;:\quad Y^\Gamma_{[3]} \;\longrightarrow\; \frac{\mathbb{C}^3}{\Gamma} \tag{1.3}$$

[1] For a recent review of these matters see chapter 8 of [18].

where $Y_{[3]}^{\Gamma}$ is a noncompact smooth three-fold with trivial canonical bundle. On such a complex three-fold a Ricci-flat Kähler metric

$$\mathrm{ds}^2_{\mathrm{RFK}}(Y_{[3]}^{\Gamma}) \;=\; \mathbf{g}^{\mathrm{RFK}}_{\alpha\beta^\star}\, dy^\alpha \otimes dy^{\beta^\star} \tag{1.4}$$

with asymptotically conical boundary conditions (Quasi-ALE) is guaranteed to exist (see e.g. [28], Thm. 3.3), *although it is not necessarily the one obtained by means of the Kähler quotient.* According to the theorem proved by Ito-Reid [19, 22, 23] and based on the concept of age grading[2], the homology cycles of $Y_{[3]}^{\Gamma}$ are all algebraic and its non vanishing cohomology groups are all even and of type $\mathrm{H}^{(q,q)}$. We actually have a correspondence between the cohomology classes of type (q,q) and the discrete group conjugacy classes with age-grading q, encoded in the statement:

$$\begin{aligned}
\dim \mathrm{H}^{1,1}\left(Y_{[3]}^{\Gamma}\right) \;&=\; \#\ \text{of junior conjugacy classes in } \Gamma;\\
\dim \mathrm{H}^{2,2}\left(Y_{[3]}^{\Gamma}\right) \;&=\; \#\ \text{of senior conjugacy classes in } \Gamma;\\
&\quad\ \text{all other cohomology groups are trivial}
\end{aligned} \tag{1.5}$$

The absence of harmonic forms of type $(2,1)$ implies that the three-folds $Y_{[3]}^{\Gamma}$ admit no infinitesimal deformations of their complex structure and is also a serious obstacle, as we discuss in section 2 to the construction of supergravity D3-brane solutions based on $Y_{[3]}^{\Gamma}$ that have transverse three-form fluxes.

There is however a possible way out that is provided by the existence of mass-deformations. This is the main point of another line of investigation which we hope to report on soon. If the McKay quiver diagram has certain properties, the superpotential $\mathcal{W}(\Phi)$ on the gauge-theory side of the correspondence can be deformed by well defined mass-terms and, after (gaussian) integration of the massive fields, the McKay quiver is remodeled into a new non-McKay quiver associated with the Kähler or HyperKähler quotient description of smooth Kähler manifolds, like the resolved conifold, that admit harmonic $(2,1)$-forms and sustain adequate D3-brane solutions.

On the basis of the above remarks we can spell-out the scope of the present paper in the following way. The embedding (1.2) determines in a unique way a McKay quiver diagram which determines:

1. the gauge group $\mathcal{F}_\Gamma$,
2. the matter field content Φ^I of the gauge theory,
3. the representation assignments of all the matter fields Φ^I,
4. the possible (mass)-deformations of the superpotential $\mathcal{W}(\phi)$,
5. the Ricci-flat metric on Y can be inferred, by means of the Monge-Ampère equation, from the Kähler metric on the exceptional compact divisor (in those cases where it exists) in the resolution of $\mathbb{C}^3/\Gamma$, which, on its turn, is determined by the McKay quiver through the Kronheimer construction.

In relation with point 4) of the above list, to be discussed in a future paper, for the case $\mathbb{C}^3/\mathbb{Z}_4$ we anticipate the following. By means of gaussian integration we get a new quiver diagram that is not directly associated with a discrete group, yet it follows from the McKay quiver of Γ in a unique way. The group theoretical approach allows us to identify deformations of the superpotential and introduce new directions in the moduli space of the crepant resolution. In this sense, we go beyond the Ito-Reid theorem. Both physically and mathematically this is quite interesting and provides a new viewpoint on several results, some of them well known in the literature. Most of the latter are based on cyclic groups Γ and rely on the powerful weapons of toric geometry. Yet the generalized Kronheimer construction applies also to non abelian groups $\Gamma \subset \mathrm{SU}(3)$ and so do the cohomological theorems proved by Ito-Reid, Ishii and Craw. Hence available mass-deformations are encoded also in the McKay quivers of non abelian groups Γ and one might explore the geometry of the transverse manifolds emerging in these cases.

[2]For a recent review of these matters within a general framework of applications to brane gauge theories see [6, 7].

In relation with point 5) of the above list, fully treated in this paper for the same case $\mathbb{C}^3/\mathbb{Z}_4$, we stress that, although the Kronheimer metric on $Y^\Gamma_{[3]}$ is not Ricci-flat, yet its restriction to the exceptional divisor provides the appropriate starting point for an iterative solution of the Monge Ampère equation which determines the Ricci-flat metric.

In view of the above considerations we can conclude that the McKay quiver diagram does indeed provide a determination of both sides of a D3-brane dual pair, the gauge theory side and the supergravity side.

In this paper we focus on two paradigmatic examples, namely $\mathbb{C}^3/\mathbb{Z}_3$ (with $\mathbb{Z}_3$ diagonally embedded in SU(3)) and $\mathbb{C}^3/\mathbb{Z}_4$. The latter case was studied in depth in a recent publication [7]. Relying on those results here we concentrate on the issue of the Ricci-flat Kähler metric.

While in the case of HyperKähler quotients (yielding $\mathcal{N}=2$ gauge theories and corresponding to the original Kronheimer construction of $\mathbb{C}^2/\Gamma$ resolutions) the *Kronheimer metric* is automatically Ricci-flat, in the case of Kähler quotients and of the generalized Kronheimer construction of $\mathbb{C}^3/\Gamma$ resolutions, the *Kronheimer metric* is not Ricci-flat and one needs to resort to different techniques in order to find a Ricci-flat metric on the same three-fold $Y^\Gamma_{[3]}$ that is algebraically determined by the construction.

The fascinating scenario that emerges from our combined analytical and numerical results is summarized in the following discussion.

From the point of view of complex algebraic geometry the resolved variety $Y^\Gamma_{[3]}$ is in many cases and, in particular in those here analyzed, the total space of a line-bundle over a compact complex two-fold, which coincides with the exceptional divisor $\mathcal{ED}$ of the resolution of singularities:

$$Y^\Gamma_{[3]} \xrightarrow{\pi} \mathcal{ED}_{[2]} \quad ; \quad \forall\, p \in \mathcal{ED}_{[2]} \quad : \quad \pi^{-1}(p) \sim \mathbb{C} \tag{1.6}$$

In the paradigmatic example, recently studied in [7], of the resolution à la Kronheimer of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity, $\mathcal{ED}$ is indeed the compact component of the exceptional divisor emerging from the blow-up of the singular point in the origin of $\mathbb{C}^3$ and it happens to be the second Hirzebruch surface $\mathbb{F}_2$. Other cases are possible.

Hirzebruch surfaces are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$, so that

$$\mathcal{ED}_{[2]} \xrightarrow{\widetilde{\pi}} \mathbb{P}^1 \quad ; \quad \forall\, p \in \mathbb{P}^1 \quad : \quad \widetilde{\pi}^{-1}(p) \sim \mathbb{P}^1 \tag{1.7}$$

This double fibration is illustrated in a pictorial fashion in fig.1.

Given this hierarchical structure, the sought for Ricci-flat metric is constrained to possess the following continuous isometry group:

$$\mathrm{G}_{iso} = \mathrm{SU(2)} \times \mathrm{U(1)_v} \times \mathrm{U(1)_w} \tag{1.8}$$

whose holomorphic algebraic action on the three coordinates u, v, w is described later in eq. (7.1). The chosen isometry group implies that the sought for Ricci-flat metric is toric, as each of the three complex coordinates is acted on by an independent U(1)-isometry. Furthermore the enhancement of one of the U(1)'s to SU(2) guarantees that either the Kähler potential $\mathcal{K}$ in the standard complex formulation of Kähler geometry, or the symplectic potential $\mathcal{G}$, the Legendre transform of the former appearing in the available symplectic formalism [60], are functions only of two invariant real variables (see section 6 and 9.1). Assuming that we possess either one of these two real functions for the Ricci-flat metric [3] :

$$\mathcal{K}_{\text{Ricci-flat}}(\varpi, \mathfrak{f}) \quad \text{or} \quad \mathcal{G}_{\text{Ricci-flat}}(\mathfrak{v}, \mathfrak{w}) \tag{1.9}$$

we can reduce the corresponding geometry to that of the exceptional divisor by setting a section of the $Y^\Gamma_{[3]}$ bundle to zero as:

$$w = 0 \quad \Leftrightarrow \quad \mathfrak{f} = 0 \,, \; \mathfrak{w} = \tfrac{3}{2} \tag{1.10}$$

The fascinating scenario we have alluded to some lines above is encoded in the following:

[3] For conventions see once again sections 9.1 and 6.

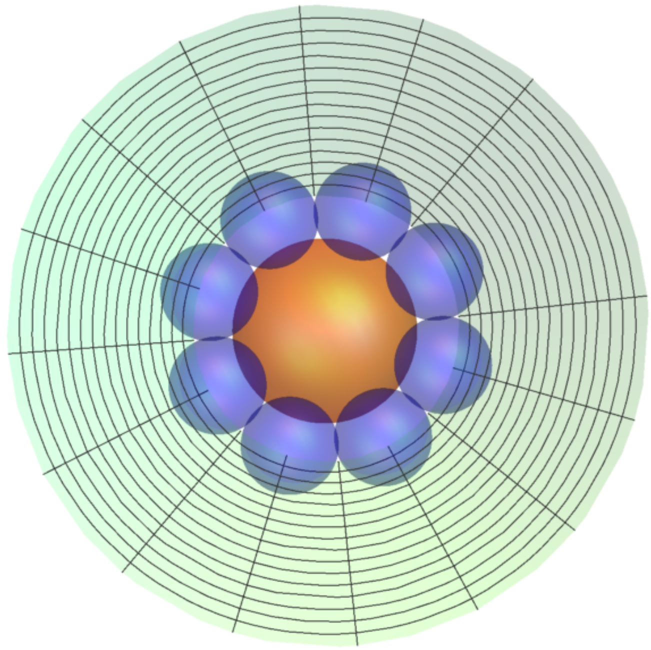

Figure 1: A conceptual picture of the resolved three-fold $Y^{\Gamma}_{[3]}$ displaying its double fribration structure. The orange sphere in the middle symbolizes the base manifold of the bundle $\mathcal{ED}_{[2]}$. A dense complex coordinate patch for this $\mathbb{P}^1$ is named u in the main body of the article. The blueish spheres around the orange one symbolize the $\mathbb{P}^1$ fibers of $\mathcal{ED}_{[2]}$. A dense complex coordinate patch for these fibers is named v in the main body of the article. Finally the greenish rays enveloping the divisor $\mathcal{ED}_{[2]}$ symbolize the noncompact fibers of the bundle $Y^{\Gamma}_{[3]}$. A dense coordinate patch for these fibers is named w in the main body of the article.

Conjecture 1.1. *The Kronheimer Kähler metric* $\mathrm{ds}^2_{Kro}[Y^{\Gamma}_{[3]}]$ *on the line bundle (1.6) and the Ricci-flat one* $\mathrm{ds}^2_{Ricflat}[Y^{\Gamma}_{[3]}]$ *on the same manifold, that has the same isometries and is asymptotically locally flat*[4] *are different, yet they coincide on the exceptional divisor* $\mathcal{ED}$*.*

The basic argument in favor of this conjecture is provided by an in depth analysis of a particular orthotoric metric that we construct in this paper and that is shown to describe the Ricci-flat metric on a degenerate limit of three-fold $Y^{\Gamma}_{[3]}$, as described in [7]. This is a partial resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity and it occurs when the stability parameters (Fayet-Iliopolous parameters in the physics jargon) are restricted to be on the unique type III wall[5] appearing in the chamber structure associated with the generalized Kronheimer construction for this McKay quiver. From the algebraic geometry viewpoint, this

[4]More precisely, this metric is Quasi-ALE in the sense of [28].

[5]According to the terminology in [23], a wall in the space of stability parameters is of type III when it corresponds to a degeneration which contracts divisors to curves. In this case the noncompact component $\mathbb{P}^1 \times \mathbb{C}$ of the exceptional divisor shrinks to $\mathbb{C}$.

variety $Y_{[3]}$ is the total space of the canonical bundle over the weighted projective space $\mathbb{W}P[112]$:

$$Y_{[3]} = \text{tot}K_{\mathbb{W}P[112]} \tag{1.11}$$

and its exceptional divisor is $\mathbb{W}P[112]$. We show that the Kähler metric induced on $\mathbb{W}P[112]$ by our new Ricci-flat orthotoric metric is precisely identical with that obtained from the Kronheimer construction once reduced to the divisor.

The various inspections of this known case within the framework of different formalisms and using different coordinate patches provided us with the means to make conjecture 1.1 more robust. The main tool at our disposal is provided by the Monge-Ampère (MA) equation for Ricci-flatness of which we develop two versions, one in terms of the Kähler potential $\mathcal{K}(\varpi, \mathfrak{f})$ (see section 9.1) and one in terms of the symplectic potential[6] $\mathcal{G}(\mathfrak{v}, \mathfrak{w})$(see section 9.2). In both cases we showed that the potential can be developed in power series of the invariant associated with the non compact fibers (either $\mathfrak{f}$ or $\mathfrak{w} - \frac{3}{2}$) and that the MA equation imposes no restriction on the 0-th order potentials $\mathcal{K}_0(\varpi)$ or $\mathcal{G}_0(\mathfrak{v})$, namely on the geometry chosen for the exceptional divisor. Rather, dealing carefully with the boundary conditions, we discovered that in both cases the MA equation completely determines all the other terms once $\mathcal{K}_0(\varpi)$ or $\mathcal{G}_0(\mathfrak{v})$ are given. Hence we can start with $\mathcal{K}_0^{Kro}(\varpi)$ or $\mathcal{G}_0^{Kro}(\mathfrak{v})$ as they are determined by the Kronheimer construction and going through the power series treatment of the MA equation we can construct a corresponding Ricci-flat metric.

The only question which remains open is whether this Ricci-flat metric is asymptotically locally flat. In the case of $\text{tot}K_{\mathbb{W}P[112]}$ it is. This supports the conjecture. In order to transform the conjecture into a theorem one should first resum the series and study the metric at large distances.

In this respect our study of the symplectic potential produced encouraging results. First of all we were able to construct an explicit form $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v}, \mathfrak{w})$ of such potential for the orthotoric case. The function $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v}, \mathfrak{w})$, which is relatively simply written in terms of elementary transcendental functions, satisfies the MA equation and can be expanded in series of $(\mathfrak{w} - \frac{3}{2})$. The remarkably similar behavior of the series truncations of the exact solution corresponding to $\text{tot}K_{\mathbb{W}P[112]}$ with the same truncations of the series determined by the MA equation for the smooth case $\text{tot}K_{\mathbb{F}_2}$ suggests that also in the latter case there exists a summation of the series to some simple deformation of the function $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v}, \mathfrak{w})$.

We postpone to future publications further attempts to sum the series solution and prove, if possible, our conjecture.

2 D3-brane supergravity solutions on resolved $\mathbb{C}^3/\Gamma$ singularities

An apparently general property of the $Y_{[3]}^{\Gamma}$ manifolds that emerge from the crepant resolution construction, at least when Γ is abelian and cyclic is the following. The non-compact $Y_{[3]}^{\Gamma}$ corresponds to the total space of some line-bundle over a complex two-dimensional compact base manifold $\mathcal{M}_2$:

$$Y_{[3]}^{\Gamma} \xrightarrow{\pi} \mathcal{M}_2 \tag{2.1}$$

According with this structure we name u, v, w the three complex coordinates of $Y_{[3]}^{\Gamma}$, u, v being the coordinates of the base manifold $\mathcal{M}_2$ and w being the coordinate spanning the fibers. We will use the same names also in more general cases even if the interpretation of w as fiber coordinate will be lost. Hence we have:

$$\boldsymbol{y} \equiv y^{\alpha} = \{u, v, w\} \quad ; \quad \overline{\boldsymbol{y}} \equiv y^{\overline{\alpha}} = \{\overline{u}, \overline{v}, \overline{w}\} \tag{2.2}$$

An important observation which ought to be done right at the beginning is that other Kähler metrics $\widehat{\mathbf{g}}_{\alpha\beta^\star}$ do exist on the three-fold $Y_{[3]}$ that are not Ricci-flat, although the cohomology class of the associated Kähler form $\widehat{\mathbf{K}}$ can be the same as the cohomology class of $\mathbf{K}_{\text{RFK}}$. Within the framework of the generalized

[6] See section 6 for the definition of the real variables $\mathfrak{v}, \mathfrak{w}$.

Kronheimer construction, among such Kähler (non-Ricci flat) metrics we have the one determined by the Kähler quotient according to the formula of Hithchin, Karlhede, Lindström and Roček [30]. Indeed, as we show later in explicit examples, the Kähler metric:

$$\mathrm{ds}^2_{\mathrm{HKLR}}(Y_{[3]}) \;=\; \mathbf{g}^{\mathrm{HKLR}}_{\alpha\beta^\star}\, dy^\alpha \otimes dy^{\beta^\star} \tag{2.3}$$

which emerges from the mathematical Kähler quotient construction and which is naturally associated with $Y_{[3]}$ when this latter is interpreted as the *space of classical vacua* of the D3-brane gauge theory (set of extrema of the scalar potential), is generically non Ricci-flat.

On the other hand on *the supergravity side* of the dual D3-brane pair we need the Ricci-flat metric in order to construct a bona-fide D3-brane solution of type IIB supergravity. In particular, calling $Y^{\Gamma}_{[3]}$ the crepant resolution of the $\mathbb{C}^3/\Gamma$ singularity, admitting a Ricci-flat metric, we can construct a bona-fide D3 brane solution which is solely defined by a single real function H on $Y^{\Gamma}_{[3]}$, that should be harmonic with respect to the Ricci-flat metric, namely:

$$\Box_{\mathbf{g}^{\mathrm{RFK}}}\, H \;=\; 0 \tag{2.4}$$

Indeed the function $H(\mathbf{y})$ is necessary and sufficient to introduce a flux of the Ramond 5-form so as to produce the splitting of the 10-dimensional space into a 4-dimensional world volume plus a transverse 6-dimensional space that is identified with the three-fold $Y^{\Gamma}_{[3]}$. This is the very essence of the D3-picture.

Yet there is another essential item that was pioneered in [31, 32, 33] namely the consistent addition of fluxes for the complex 3-forms $\mathcal{H}_\pm$ that appear in the field content of type IIB supergravity. These provide relevant new charges on both sides of the gauge/gravity correspondence. In [34, 35] such fluxes were constructed explicitely relying on a special kind of three-fold:

$$Y_{[3]} \;=\; Y_{[1+2]} \;=\; \mathbb{C} \times \mathrm{ALE}_\Gamma \tag{2.5}$$

where ALE_Γ denotes one of the ALE-manifolds constructed by Kronheimer [15, 16] as HyperKähler quotients resolving the singularity $\mathbb{C}^2/\Gamma$ with $\Gamma \subset \mathrm{SU}(2)$ a finite Kleinian subgroup.

As we explain in detail below, the essential geometrical feature of $Y_{[3]}$, required to construct consistent fluxes of the complex 3-forms $\mathcal{H}_\pm$, is that $Y_{[3]}$ should admit imaginary *(anti)-self-dual, harmonic* 3*-forms* $\Omega^{(2,1)}$, which means:

$$\star_{\mathbf{g}^{\mathrm{RFK}}}\, \Omega^{(2,1)} \;=\; \pm\,\mathrm{i}\,\Omega^{(2,1)} \tag{2.6}$$

and simultaneously:

$$d\Omega^{(2,1)} \;=\; 0 \quad \Rightarrow \quad d\star_{\mathbf{g}^{\mathrm{RFK}}}\Omega^{(2,1)} \;=\; 0 \tag{2.7}$$

Since the Hodge-duality operator involves the use of a metric, we have been careful in specifying that (anti)-self-duality should occur with respect to the Ricci-flat metric that is the one used in the rest of the supergravity solution construction.

The reason why the choice (2.5) of the three-fold allows the existence of harmonic anti-self dual 3-forms is easily understood recalling that the ALE_Γ-manifold obtained from the resolution of $\mathbb{C}^2/\Gamma$ has a compact support cohomology group of type $(1,1)$ of the following dimension:

$$\dim \mathrm{H}^{(1,1)}_{comp}\,(\mathrm{ALE}_\Gamma) \;=\; r \quad \text{where} \quad r \;=\; \#\; \text{of nontrivial conjugacy classes of } \Gamma \tag{2.8}$$

Naming $z \in \mathbb{C}$ the coordinate on the factor $\mathbb{C}$ of the product (2.5) and $\omega^{(1,1)}_I$ a basis of harmonic anti-self dual one-forms on ALE_Γ, the ansatz utilized in [34, 35] to construct the required $\Omega^{(2,1)}$ was the following:

$$\Omega^{(2,1)} \;\equiv\; \partial_z \mathfrak{f}^I(z)\, dz \,\wedge\, \omega^{(1,1)}_I \tag{2.9}$$

where $\mathfrak{f}^I(z)$ is a set of holomorphic functions of that variable. As it is well known r is also the rank of the corresponding Lie Algebra in the ADE-classification of the corresponding Kleinian groups and the 2-forms $\omega^{(1,1)}_I$ can be chosen dual to a basis homology cycles $\mathcal{C}_I$ spanning $H_2\,(\mathrm{ALE}_\Gamma)$, namely we can set:

$$\int_{\mathcal{C}_I} \omega^{(1,1)}_J \;=\; \delta_{IJ} \tag{2.10}$$

The form $\Omega^{(2,1)}$ is closed by construction:

$$d\Omega^{(2,1)} = 0 \tag{2.11}$$

and it is also anti-selfdual with respect to the Ricci-flat metric:

$$\mathrm{ds}^2_{Y_{[1+2]}} = dz \otimes d\bar{z} + \mathrm{ds}^2_{\mathrm{ALE}_\Gamma} \tag{2.12}$$

Hence the question whether we can construct sufficiently flexible D3-solutions of supergravity with both 5-form and 3-form fluxes depends on the nontriviality of the relevant cohomology group:

$$\dim \mathrm{H}^{(2,1)}\left(Y_{[3]}\right) > 0 \tag{2.13}$$

and on our ability to find harmonic (anti)-self dual representatives of its classes (typically not with compact support and hence non normalizable).

At this level we find a serious difficulty. It seems therefore that we are not able to find the required $\Omega^{(2,1)}$ forms on $Y^{\Gamma}_{[3]}$ and that no D3-brane supergravity solution with 3-form fluxes can be constructed dual to the gauge theory obtained from the Kronheimer construction dictated by $\Gamma \subset \mathrm{SU}(3)$. Fortunately, the sharp conclusion encoded in eq. (1.5) follows from a hidden mathematical assumption that, in physical jargon, amounts to a rigid universal choice of the holomorphic superpotential $\mathcal{W}(\Phi)$. Under appropriate conditions that we plan to explain and which are detectable at the level of the McKay quiver diagram, the superpotential can be deformed (mass deformation) yielding a family of three-folds $Y^{\Gamma,\mu}_{[3]}$ which flow, for limiting values of the parameter ($\mu \to \mu_0$) to a three-fold $Y^{\Gamma,\mu_0}_{[3]}$ admitting imaginary anti self-dual harmonic (2,1)-forms. Since the content and the interactions of the gauge theory are dictated by the McKay quiver of Γ and by its associated Kronheimer construction, we are entitled to see its mass deformed version and the exact D3-brane supergravity solution built on $Y^{\Gamma,\mu_0}_{[3]}$ as dual to each other.

This will be the object of a future work. Here we begin with an accurate mathematical summary of the construction of D3-brane solutions of type IIB supergravity using the geometric formulation of the latter within the rheonomy framework [36].

2.1 Geometric formulation of Type IIB supergravity

In order to discuss conveniently the D3 brane solutions of type IIB that have as transverse space the crepant resolution of a $\mathbb{C}^3/\Gamma$ singularity, we have to recall the geometric Free Differential Algebra formulation of the chiral ten dimensional theory fixing with care all our conventions, which is not only a matter of notations but also of principles and geometrical insight. Indeed the formulation of type IIB supergravity as it appears in string theory textbooks [37, 38] is tailored for the comparison with superstring amplitudes and is quite appropriate to this goal. Yet, from the viewpoint of the general geometrical set up of supergravity theories this formulation is somewhat unwieldy. Specifically it neither makes the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset structure of the theory manifest, nor does it relate the supersymmetry transformation rules to the underlying algebraic structure which, as in all other instances of supergravities, is a simple and well defined *Free Differential algebra*. The Free Differential Algebra of type IIB supergravity was singled out many years ago by Castellani in [39] and the geometric, manifestly $\mathrm{SU}(1,1)$–covariant formulation of the theory was constructed by Castellani and Pesando in [40]. Their formulae and their transcription from a complex $\mathrm{SU}(1,1)$ basis to a real $\mathrm{SL}(2,\mathbb{R})$ basis were summarized and thoroughly explained in a dedicated chapter of a book authored by one of us [41] which we refer the reader to.

2.2 The D3-brane solution with a $Y_{[3]}$ transverse manifold

In this section we discuss a D3-brane solution of type IIB supergravity in which, transverse to the brane world-manifold, we place a smooth non compact three-fold $Y_{[3]}$ endowed with a Ricci-flat Kähler metric.

The ansatz for the D3-brane solution is characterized by two kinds of flux; in addition to the usual RR 5-form flux, there is a non-trivial flux of the supergravity complex 3-form field strengths $\mathcal{H}_{\pm}$.

We separate the ten coordinates of space-time into the following subsets:

$$x^M = \begin{cases} x^\mu & : \quad \mu = 0,1,2,3 \qquad \text{coordinates of the 3-brane world volume} \\ y^\tau & : \quad \tau = 4,5,6,7,8,9 \quad \text{real coordinates of the } Y \text{ variety} \end{cases} \tag{2.1}$$

2.2.1 The D3 brane ansatz

We make the following ansatz for the metric[7]:

$$\begin{aligned} \mathrm{ds}^2_{[10]} &= H(\boldsymbol{y},\overline{\boldsymbol{y}})^{-\frac{1}{2}} \left(-\eta_{\mu\nu}\, dx^\mu \otimes dx^\nu\right) + H(\boldsymbol{y},\overline{\boldsymbol{y}})^{\frac{1}{2}} \left(\mathbf{g}^{\mathrm{RFK}}_{\alpha\beta^\star}\, dy^\alpha \otimes dy^{\beta^\star}\right) \\ \mathrm{ds}^2_Y &= \mathbf{g}^{\mathrm{RFK}}_{\alpha\beta^\star}\, dy^\alpha \otimes dy^{\beta^\star} \\ \det(g_{[10]}) &= H(\boldsymbol{y},\overline{\boldsymbol{y}}) \det(\mathbf{g}^{\mathrm{RFK}}) \\ \eta_{\mu\nu} &= \mathrm{diag}(+,-,-,-) \end{aligned} \tag{2.2}$$

where $\mathbf{g}^{\mathrm{RFK}}$ is the Kähler metric of the $Y_{[3]}$ manifold

$$\mathbf{g}^{\mathrm{RFK}}_{\alpha\overline{\beta}} = \partial_\alpha\, \partial_{\overline{\beta}}\, \mathcal{K}^{\mathrm{RFK}}\left(\boldsymbol{y},\overline{\boldsymbol{y}}\right) \tag{2.3}$$

the real function $\mathcal{K}^{\mathrm{RFK}}\left(\boldsymbol{y},\overline{\boldsymbol{y}}\right)$ being a suitable Kähler potential.

2.2.2 Elaboration of the ansatz

In terms of vielbein the ansatz (2.2) corresponds to

$$V^A = \begin{cases} V^a &= H(\boldsymbol{y},\overline{\boldsymbol{y}})^{-1/4}\, dx^a \qquad a = 0,1,2,3 \\ V^\ell &= H(\boldsymbol{y},\overline{\boldsymbol{y}})^{1/4}\, \mathbf{e}^\ell \qquad \ell = 4,5,6,7,8,9 \end{cases} \tag{2.4}$$

where $\mathbf{e}^\ell$ are the vielbein 1-forms of the manifold $Y_{[3]}$. The structure equations of the latter are[8]:

$$\begin{aligned} 0 &= d\,\mathbf{e}^i - \widehat{\omega}^{ij} \wedge \mathbf{e}^k\, \eta_{jk} \\ \widehat{R}^{ij} &= d\widehat{\omega}^{ij} - \widehat{\omega}^{ik} \wedge \widehat{\omega}^{\ell j}\, \eta_{k\ell} = \widehat{R}^{ij}{}_{\ell m}\, \mathbf{e}^\ell \wedge \mathbf{e}^m \end{aligned} \tag{2.5}$$

The relevant property of the Y metric that we use in solving Einstein equations is that it is Ricci-flat:

$$\widehat{R}^{im}{}_{\ell m} = 0 \tag{2.6}$$

What we need in order to derive our solution and discuss its supersymmetry properties is the explicit form of the spin connection for the full 10-dimensional metric (2.2) and the corresponding Ricci tensor. From the torsion equation one can uniquely determine the solution:

$$\begin{aligned} \omega^{ab} &= 0 \\ \omega^{a\ell} &= \frac{1}{4}\, H^{-3/2}\, dx^a \eta^{\ell k}\, \partial_k\, H \\ \omega^{\ell m} &= \widehat{\omega}^{\ell m} + \Delta\omega^{\ell m} \quad ; \quad \Delta\omega^{\ell m} = -\frac{1}{2}\, H^{-1}\, \mathbf{e}^{[\ell}\, \eta^{m]k}\, \partial_k H \end{aligned} \tag{2.7}$$

[7]As explained in appendix A, the conventions for the gamma matrices and the spinors are set with a mostly minus metric $d\tau^2$. In the discussion of the solution, however, we use $ds^2 = -d\tau^2$ for convenience. We hope this does not cause any confusion.

[8]The hats over the spin connection and the Riemann tensor denote quantities computed without the warp factor.

Inserting this result into the definition of the curvature 2-form we obtain[9]:

$$\begin{aligned}
R^a_b &= -\frac{1}{8}\left[H^{-3/2}\Box_{\mathbf{g}}\,H - H^{-5/2}\,\partial_k H\partial^k H\right]\,\delta^a_b \\
R^a_\ell &= 0 \\
R^m_\ell &= \frac{1}{8}H^{-3/2}\Box_{\mathbf{g}}H\delta^m_\ell - \frac{1}{8}H^{-5/2}\partial_s H\partial^s H\delta^m_\ell + \frac{1}{4}H^{-5/2}\partial_\ell H\partial^m H
\end{aligned} \tag{2.8}$$

where for any function $f\,(\boldsymbol{y},\overline{\boldsymbol{y}})$ with support on $Y_{[3]}$:

$$\Box_{\mathbf{g}}\,f\,(\boldsymbol{y},\overline{\boldsymbol{y}}) \;=\; \frac{1}{\sqrt{\det\mathbf{g}}}\,\left(\partial_\alpha\left(\sqrt{\det\mathbf{g}}\;\mathbf{g}^{\alpha\beta^\star}\,\partial_{\beta^\star}\,f\right)\right) \tag{2.9}$$

denotes the action on it of the Laplace–Beltrami operator with respect to the metric (2.3) which is the Ricci-flat one: we have omitted the superscript RFK just for simplicity. Indeed on the supergravity side of the correspondence we use only the Ricci-flat metric and there is no ambiguity.

2.2.3 Analysis of the field equations in geometrical terms

The equations of motion for the scalar fields φ and $C_{[0]}$ and for the 3-form field strength $F^{NS}_{[3]}$ and $F^{RR}_{[3]}$ can be better analyzed using the complex notation. Defining, as we did above:

$$\begin{aligned}
\mathcal{H}_\pm &= \pm 2\,e^{-\varphi/2}F^{NS}_{[3]} + \mathrm{i}2\,e^{\varphi/2}\,F^{RR}_{[3]} &(2.10)\\
P &= \tfrac{1}{2}\,d\varphi - \mathrm{i}\tfrac{1}{2}\,e^{\varphi}\,F^{RR}_{[1]} &(2.11)
\end{aligned}$$

eq.s (2.12)-(2.13) can be respectively written as:

$$d(\star P) - \mathrm{i}e^{\varphi}dC_{[0]}\wedge\star P + \tfrac{1}{16}\mathcal{H}_+\wedge\star\mathcal{H}_+ = 0 \tag{2.12}$$

$$d\star\mathcal{H}_+ - \frac{\mathrm{i}}{2}e^{\varphi}dC_{[0]}\wedge\star\mathcal{H}_+ = \mathrm{i}\,F^{RR}_{[5]}\wedge\mathcal{H}_+ - P\wedge\star\mathcal{H}_- \tag{2.13}$$

while the equation for the 5-form becomes:

$$d\star F^{RR}_{[5]} = \mathrm{i}\,\tfrac{1}{8}\,\mathcal{H}_+\wedge\mathcal{H}_- \tag{2.14}$$

Besides assuming the structure (2.2) we also assume that the two scalar fields, namely the dilaton φ and the Ramond-Ramond 0-form $C_{[0]}$ are constant and vanishing:

$$\varphi = 0 \quad ; \quad C_{[0]} = 0 \tag{2.15}$$

As we shall see, this assumption simplifies considerably the equations of motion, although these two scalar fields can be easily restored [33].

[9] The reader should be careful with the indices. Latin indices are always frame indices referring to the vielbein formalism. Furthermore we distinguish the 4 directions of the brane volume by using Latin letters from the beginning of the alphabet while the 6 transversal directions are denoted by Latin letters from the middle and the end of the alphabet. For the coordinate indices we utilize Greek letters and we do exactly the reverse. Early Greek letters $\alpha,\beta,\gamma,\delta,\ldots$ refer to the 6 transverse directions while Greek letters from the second half of the alphabet $\mu,\nu,\rho,\sigma,\ldots$ refer to the D3 brane world volume directions as it is customary in $D=4$ field theories.

2.2.4 The three-forms

The basic ansatz characterizing the solution and providing its interpretation as a D3-brane with three-form fluxes is described below.

The ansatz for the complex three-forms of type IIB supergravity is given below and is inspired by what was done in [35, 34] in the case where $Y_{[3]} = \mathbb{C} \times \mathrm{ALE}_\Gamma$:

$$\mathcal{H}_+ \;=\; \Omega^{(2,1)} \tag{2.16}$$

where $\Omega^{(2,1)}$ is localized on $Y_{[3]}$ and satisfies eq.s (2.6-2.7)

If we insert the ansätze (2.15,2.16) into the scalar field equation (2.12) we obtain:

$$\mathcal{H}_+ \wedge \star_{10} \mathcal{H}_+ = 0 \tag{2.17}$$

This equation is automatically satisfied by our ansatz for a very simple reason that we explain next. The form $\mathcal{H}_+$ is by choice a three-form on $Y_{[3]}$ of type $(2,1)$. Let $\Theta^{[3]}$ be any three-form that is localized on the transverse six-dimensional [10] manifold $Y_{[3]}$:

$$\Theta^{[3]} \;=\; \Theta_{IJK}\, dt^I \wedge dt^J \wedge dt^K \tag{2.18}$$

When we calculate the Hodge dual of $\Theta^{[3]}$ with respect to the 10-dimensional metric (2.2) we obtain a 7-form with the following structure:

$$\star_{10}\; \Theta^{[3]} \;=\; H^{-1}\, \mathrm{Vol}_{\mathbb{R}^{(1,3)}} \;\wedge\; \widetilde{\Theta}^{[3]} \tag{2.19}$$

where:

$$\mathrm{Vol}_{\mathbb{R}^{(1,3)}} \;=\; \tfrac{1}{4!}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\, \epsilon_{\mu\nu\rho\sigma} \tag{2.20}$$

is the volume-form of the flat D3-brane and

$$\widetilde{\Theta}^{[3]} \;\equiv\; \star_{\mathbf{g}}\, \Theta^{[3]} \tag{2.21}$$

is the dual of the three-form $\Theta^{[3]}$ with respect to the metric $\mathbf{g}$ defined on $Y_{[3]}$. Let us now specialize the three-form $\Theta^{[3]}$ to be of type $(2,1)$:

$$\Theta^{[3]} \;=\; \mathrm{Q}^{(2,1)} \tag{2.22}$$

As shown in [31, 32], preservation of supersymmetry requires the complex three-form $\mathcal{H}_+$ to obey the condition[11]

$$\star_{\mathbf{g}}\; \mathrm{Q}^{(2,1)} \;=\; -\mathrm{i}\, \mathrm{Q}^{(2,1)} \tag{2.23}$$

Hence:

$$\mathcal{H}_+ \wedge \star_{10}\, \mathcal{H}_+ \;=\; -\mathrm{i}\, \mathrm{Q}^{(2,1)} \;\wedge\; \mathrm{Q}^{(2,1)} \;\wedge\; H^{-1}\mathrm{Vol}_{\mathbb{R}^{(1,3)}} \;=\; 0 \tag{2.24}$$

2.2.5 The self-dual 5-form

Next we consider the self-dual 5-form $F^{RR}_{[5]}$ which by definition must satisfy the following Bianchi identity:

$$d\, F^{RR}_{[5]} = \mathrm{i}\, \tfrac{1}{8}\, \mathcal{H}_+ \;\wedge\; \mathcal{H}_- \tag{2.25}$$

[10] For the sake of the present calculation and the following ones where we have to calculate a Hodge dual, it is more convenient to utilize a set of 6 real coordinates t^I $(I = 1, \ldots, 6)$ for the manifold $Y_{[3]}$. Let $\partial_I \equiv \frac{\partial}{\partial t^I}$ denote the standard partial derivatives with respect to such coordinates.

[11] It also requires $\mathcal{H}_+$ to be primitive.

Our ansatz for $F^{RR}_{[5]}$ is the following:

$$F^{RR}_{[5]} = \alpha \left(U + \star_{10} U\right) \tag{2.26}$$
$$U = d\left(H^{-1}\,\mathrm{Vol}_{\mathbb{R}^{(1,3)}}\right) \tag{2.27}$$

where α is a constant to be determined later. By construction $F^{RR}_{[5]}$ is self-dual and its equation of motion is trivially satisfied. What is not guaranteed is that also the Bianchi identity (2.25) is fulfilled. Imposing it, results into a differential equation for the function $H(\boldsymbol{y},\overline{\boldsymbol{y}})$. Let us see how this works.

Starting from the ansatz (2.27) we obtain:

$$U = -\frac{1}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge \frac{dH}{H^2} \tag{2.28}$$
$$U_{\mu\nu\rho\sigma I} = -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\,\frac{\partial_I H}{H^2} \quad ; \quad \text{all other components vanish} \tag{2.29}$$

Calculating the components of the dual form $\star_{10} U$ we find that they are non vanishing uniquely in the six transverse directions:

$$\begin{aligned}
\star_{10} U &= \widetilde{U}_{I_1\dots I_5}\; dt^{I_1} \wedge \cdots \wedge dt^{I_5} \\
\widetilde{U}_{I_1\dots I_5} &= -\frac{\sqrt{\det g_{(10)}}}{5!}\epsilon_{I_1\dots I_5 J}\,\epsilon_{\mu\nu\rho\sigma}\,g^{JK}_{(10)}\,g^{\mu\mu'}_{10}\,g^{\nu\nu'}_{(10)}\,g^{\rho\rho'}_{(10)}\,g^{\sigma\sigma'}_{(10)}\,U_{\mu'\nu'\rho'\sigma' J} \\
&= \frac{\sqrt{\det\mathbf{g}}}{5!}\,\epsilon_{I_1\dots I_5 J}\,\mathbf{g}^{JK}\,\partial_K H
\end{aligned} \tag{2.30}$$

The essential point in the above calculation is that all powers of the function H exactly cancel so that $\star_{10} U$ is linear in the H-derivatives [12]. Next using the same coordinate basis we obtain:

$$\begin{aligned}
d\,F^{RR}_{[5]} &= \alpha\, d\star U = \alpha\,\underbrace{\frac{1}{\sqrt{\det\mathbf{g}}}\,\partial_I\left(\sqrt{\det\mathbf{g}}\,\mathbf{g}^{IJ}\,\partial_J H\right)}_{\Box_{\mathbf{g}}\,H} \times \mathrm{Vol}_{Y_{[3]}} \\
&= \alpha\,\Box_{\mathbf{g}}\,H(\boldsymbol{y},\overline{\boldsymbol{y}})\;\times\;\mathrm{Vol}_{Y_{[3]}}
\end{aligned} \tag{2.31}$$

where:

$$\begin{aligned}
\mathrm{Vol}_{Y_{[3]}} &\equiv \sqrt{\det\mathbf{g}}\,\frac{1}{6!}\epsilon_{I_1\dots I_6}dt^{I_1} \wedge \cdots \wedge dt^{I_6} \\
&= \sqrt{\det\mathbf{g}}\,\frac{1}{(3!)^2}\,\epsilon_{\alpha\beta\gamma}\,dy^\alpha \wedge dy^\beta \wedge dy^\gamma \wedge \epsilon_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\,d\overline{y}^{\overline{\alpha}} \wedge d\overline{y}^{\overline{\beta}} \wedge d\overline{y}^{\overline{\gamma}}
\end{aligned} \tag{2.32}$$

is the volume form of the transverse six-dimensional space. Once derived with the use of real coordinates, the relation (2.31) can be transcribed in terms of complex coordinates and the Laplace-Beltrami operator $\Box_{\mathbf{g}}$ can be written as in eq. (2.9). Let us now analyze the source terms provided by the three-forms. With our ansatz we obtain:

$$\begin{aligned}
\tfrac{1}{8}\,\mathcal{H}_+ \wedge \mathcal{H}_- &= \mathbb{J}\left(\boldsymbol{y},\overline{\boldsymbol{y}}\right)\;\times\;\mathrm{Vol}_{Y_{[3]}} \\
\mathbb{J}\left(\boldsymbol{y},\overline{\boldsymbol{y}}\right) &= -\frac{1}{72\sqrt{\det\mathbf{g}}}\;\Omega_{\alpha\beta\overline{\eta}}\;\overline{\Omega}_{\overline{\delta}\overline{\theta}\gamma}\;\epsilon^{\alpha\beta\gamma}\;\epsilon^{\overline{\eta}\overline{\delta}\overline{\theta}}
\end{aligned} \tag{2.33}$$

we conclude that the Bianchi identity (2.25) is satisfied by our ansatz if:

$$\Box_{\mathbf{g}}\,H = -\frac{1}{\alpha}\,\mathbb{J}\left(\boldsymbol{y},\overline{\boldsymbol{y}}\right) \tag{2.34}$$

This is the main differential equation to which the entire construction of the D3-brane solution can be reduced to. We are going to show that the parameter α is determined by Einstein's equations and fixed to $\alpha = 1$.

[12] Note that we use $\mathbf{g}_{IJ}$ to denote the components of the Kähler metric (2.3) in the real coordinate basis t^I.

2.2.6 The equations for the three–forms

Let us consider next the field equation for the complex three-form, namely eq. (2.13). Since the two scalar fields are constant the $\mathrm{SU}(1,1)/\mathrm{O}(2)$ connection vanishes and we have:

$$d \star \mathcal{H}_+ = \mathrm{i}\, F^{RR}_{[5]} \wedge \mathcal{H}_+ \tag{2.35}$$

Using our ansatz we immediately obtain:

$$\begin{aligned} d \star \mathcal{H}_+ = &= -2\,\mathrm{i}H^{-2} dH \wedge \widetilde{\Omega}^{(2,1)}, \wedge \Omega_{\mathbb{R}^{1,3}} + 2\mathrm{i}\, H^{-1}\, d\widetilde{\Omega}^{(2,1)} \wedge \Omega_{\mathbb{R}^{1,3}} \\ \mathrm{i}\, F^{RR}_{[5]} \wedge \mathcal{H}_+ &= -2\,\alpha \mathrm{i} H^{-2} dH \wedge \Omega^{(2,1)} \wedge \Omega_{\mathbb{R}^{1,3}} \end{aligned} \tag{2.36}$$

Hence if $\alpha = 1$, the field equations for the three-form reduces to:

$$\widetilde{\Omega}^{(2,1)} \equiv \star_{\mathbf{g}} \Omega^{(2,1)} = -\,\mathrm{i}\, \Omega^{(2,1)} \quad ; \quad d \star_{\mathbf{g}} \Omega^{(2,1)} = 0 \quad ; \quad d\Omega^{(2,1)} = 0 \tag{2.37}$$

which are nothing else but eq.s (2.6-2.7). In other words the solution of type IIB supergravity with three-form fluxes exists if and only if the transverse space admits *closed and imaginary anti-self-dual forms* $\Omega^{(2,1)}$ as we already stated[13].

In order to show that also the Einstein's equation is satisfied by our ansatz we have to calculate the (trace subtracted) stress energy tensor of the five and three index field strengths. For this purpose we need the components of $F^{RR}_{[5]}$. These are easily dealt with. Relying on the ansatz (2.27) and on eq. (2.4) for the vielbein we immediately get:

$$F_{A_1 \dots A_5} = \begin{cases} F_{iabcd} &= \frac{1}{5!}\, f_i\, \epsilon_{abcd} \\ F_{j_1 \dots i_5} &= \frac{1}{5!} \epsilon_{ij_1 \dots j_5}\, f^i \\ \text{otherwise} &= 0 \end{cases} \tag{2.38}$$

where:

$$f_i = -\alpha\, H^{-5/4}\, \partial_i H \tag{2.39}$$

Then by straightforward algebra we obtain:

$$\begin{aligned} T^a_b \left[F^{RR}_{[5]} \right] &\equiv -75\, F^{a \cdots}\, F_{b \cdots} = -\frac{1}{8}\, \delta^a_b\, f_p\, f^p \\ &= -\alpha^2\, \frac{1}{8}\, \delta^a_b\, H^{-5/2} \partial_p H\, \partial^p H \\ T^i_j \left[F^{RR}_{[5]} \right] &\equiv -75\, F^{i \cdots}\, F_{j \cdots} = \frac{1}{4}\, f^i\, f_j - \frac{1}{8}\, \delta^i_j\, f_p\, f^p \\ &= \alpha^2 \left[\frac{1}{4}\, H^{-5/4} \partial^i H\, \partial_j H - \frac{1}{8}\, \delta^i_j\, H^{-5/4} \partial^p H\, \partial_p H \right] \end{aligned} \tag{2.40}$$

Inserting eq.s (2.40) and (2.8) into Einstein's equations:

$$\begin{aligned} R^a_b &= T^a_b \left[F^{RR}_{[5]} \right] \\ R^i_j &= T^i_j \left[F^{RR}_{[5]} \right] \end{aligned} \tag{2.41}$$

we see that they are satisfied, provided

$$\alpha = 1 \tag{2.42}$$

and the master equation (2.34) is satisfied. This concludes our proof that an exact D3-brane solution with a Y transverse space does indeed exist.

[13] By construction a closed anti-self-dual form is also coclosed, namely it is harmonic.

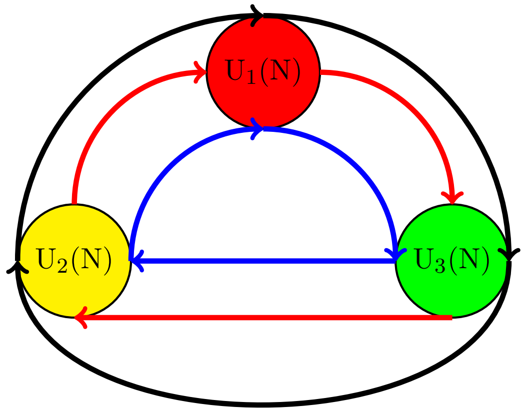

Figure 2: The quiver diagram of the diagonal embedding of the group $\mathbb{Z}_3 \to \mathrm{SU}(3)$

3 An example without mass deformations and no harmonic $\Omega^{(2,1)}$: $Y_{[3]} = \mathcal{O}_{\mathbb{P}^2}(-3)$

In [6] as a master example of the generalized Kronheimer construction of crepant resolutions the following case was considered:

$$Y_{[3]} = \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \frac{\mathbb{C}^3}{\mathbb{Z}_3} \tag{3.1}$$

the action of the group $\mathbb{Z}_3 \subset \mathrm{SU}(3)$ on the three-complex coordinates $\{x, y, z\}$ being generated by the matrix:

$$\mathfrak{g} = \begin{pmatrix} e^{\frac{2i\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{2i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{2i\pi}{3}} \end{pmatrix} \tag{3.2}$$

Following the steps of the construction one arrives at the following nine-dimensional flat Kähler manifold

$$\mathcal{S}_{\mathbb{Z}_3} \equiv \mathrm{Hom}\left(\mathcal{Q} \otimes R, R\right)^{\mathbb{Z}_3} \tag{3.3}$$

where $\mathcal{Q}$ is the three dimensional representation of $\mathbb{Z}_3$ generated by $\mathfrak{g}$, while R denotes the regular representation. The points of $\mathcal{S}_{\mathbb{Z}_3}$ are identified with the following triplet of matrices of 3×3 matrices:

$$A = \begin{pmatrix} 0 & 0 & \Phi^A_{1,3} \\ \Phi^A_{2,1} & 0 & 0 \\ 0 & \Phi^A_{3,2} & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & 0 & \Phi^B_{1,3} \\ \Phi^B_{2,1} & 0 & 0 \\ 0 & \Phi^B_{3,2} & 0 \end{pmatrix} \quad ; \quad C = \begin{pmatrix} 0 & 0 & \Phi^C_{1,3} \\ \Phi^C_{2,1} & 0 & 0 \\ 0 & \Phi^C_{3,2} & 0 \end{pmatrix} \tag{3.4}$$

The nine complex coordinates of $\mathcal{S}_{\mathbb{Z}_3}$ are the matrix entries $\Phi^{A,B,C}_{1,3}$, $\Phi^{A,B,C}_{2,1}$, $\Phi^{A,B,C}_{3,2}$. With reference to the quiver diagram of fig.2 which is dictated by the McKay matrix $\mathcal{A}_{ij}$ appearing in the decomposition [14]:

$$\mathcal{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{3} \mathcal{A}_{ij}\, \mathrm{D}_j \tag{3.5}$$

[14] D_i denote the irreducible representations of the group $\Gamma = \mathbb{Z}_3$ and each node of the quiver diagram corresponds to one of them. The number of lines going from node i to node j is equal to integer value of $\mathcal{A}_{ij}$. In each node i we have a component $U_i(n_i \times N)$ of the gauge group $\mathcal{F}_\Gamma$ where n_i is the dimension of the irrep D_i and N is the number of D3-branes.

the entries $\Phi^{A,B,C}_{1,3}, \dots$ are interpreted as the complex scalar fields of as many Wess-Zumino multiplets in the bifundamental of the $\mathrm{U_i(N)}$ groups mentioned in the lower suffix.

In the case of a single brane (N=1) the quiver group $\mathcal{G}_{\mathbb{Z}_3}$ has the following structure:

$$\mathcal{G}_{\mathbb{Z}_3} \,=\, \mathbb{C}^\star \otimes \mathbb{C}^\star \,\simeq\, \frac{\mathbb{C}^\star \otimes \mathbb{C}^\star \otimes \mathbb{C}^\star}{\mathbb{C}^\star_{central}} \tag{3.6}$$

and its maximal compact subgroup $\mathcal{F}_{\mathbb{Z}_3} \subset \mathcal{G}_{\mathbb{Z}_3}$ is the following:

$$\mathcal{F}_{\mathbb{Z}_3} \,=\, \mathrm{U}(1) \otimes \mathrm{U}(1) \,\simeq\, \frac{\mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \mathrm{U}(1)}{\mathrm{U}(1)_{central}} \tag{3.7}$$

The gauge group $\mathcal{F}_{\mathbb{Z}_3}$ and its complexification $\mathcal{F}_{\mathbb{Z}_3}$ are embedded into $\mathrm{SL}(3,\mathbb{C})$ by defining the following two generators:

$$\mathbf{t}_1 \,=\, \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \mathbf{t}_2 \,=\, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix} \tag{3.8}$$

and setting:

$$\mathcal{F}_{\mathbb{Z}_3} \,=\, \exp\left[\theta_1 \, \mathbf{t}_1 + \theta_2 \, \mathbf{t}_2\right] \quad \theta_{1,2} \in [0, 2\pi] \quad ; \quad \mathcal{G}_{\mathbb{Z}_3} \,=\, \exp\left[w_1 \, \mathbf{t}_1 + w_2 \, \mathbf{t}_2\right] \quad w_{1,2} \in \mathbb{C} \tag{3.9}$$

3.1 The HKLR Kähler potential

The Kähler potential of the linear space $\mathcal{S}_{\mathbb{Z}_3}$, which in the D3-brane gauge theory provides the kinetic terms of the nine scalar fields $\Phi^{A,B,C}_{1,2,3}$ is given by:

$$\mathcal{K}_0\,(\Phi) \,=\, \mathrm{Tr}\left(A^\dagger \, A + B^\dagger \, B + C^\dagger C\right) \tag{3.10}$$

where the three matrices A, B, C are those of equation (3.4). According with the principles of the Kronheimer construction, the superpotential is given by $\mathcal{W}\,(\Phi) \,=\, \mathrm{const} \times \mathrm{Tr}\,([A, B]\; C)$. The final HKLR Kähler metric, whose determination requires two steps of physical significance:

1. Reduction to the critical surface of the superpotential *i.e.* $\partial_\Phi \, \mathcal{W} \,=\, 0$

2. Reduction to the level surfaces of the gauge group moment maps by solving the algebraic moment map equations,

was calculated in [6] according with the general theory there summarized, which is originally due to the authors of [30]. The final form of HKLR Kähler potential is provided by:

$$\begin{aligned} \mathcal{K}_{HKLR}(z, \overline{z}, \boldsymbol{\zeta}) \;&=\; \mathcal{K}_0 + \zeta_I \, \mathfrak{C}^{IJ} \, \log\left[\Upsilon_J^{\alpha_J, \boldsymbol{\zeta}}\right] \\ &=\; \alpha\Big\{ (2\zeta_1 - \zeta_2) \log\left[\Upsilon_1\right] - (\zeta_1 - 2\zeta_2) \log\left[\Upsilon_2\right] \Big\} + \frac{\Sigma\left(\Upsilon_1^3 + \Upsilon_2^3 + 1\right)}{\Upsilon_1 \Upsilon_2} \end{aligned} \tag{3.11}$$

where, as it was extensively discussed in [7], the coefficient α might be adjusted, chamber by chamber, in chamber space, so as to make the periods of the tautological line bundles integer on the homology basis.

Setting:

$$\Sigma \,\equiv\, |z_1|^2 + |z_2|^2 + |z_3|^2 \quad ; \quad \Upsilon_{1,2} \,=\, \Upsilon_{1,2} \,=\, \Lambda_{1,2}\,(\Sigma, \boldsymbol{\zeta}) \tag{3.12}$$

where $z_{1,2,3}$ are the three complex coordinates and $\boldsymbol{\zeta} = \{\zeta_1, \zeta_2\}$ the two Fayet-Iliopoulos parameters. Let us describe the explicit form of these functions. To this effect let us name $\zeta_1 = p$, $\zeta_2 = q$, and let us introduce the following blocks:

$$\begin{aligned} \mathfrak{A} &= \sqrt{p^6\left(\left(2p^3q^3+9p^2q\Sigma^3+9pq^2\Sigma^3+27\Sigma^6\right)^2-4\left(p^2q^2+3p\Sigma^3+3q\Sigma^3\right)^3\right)} \\ \mathfrak{B} &= 2p^6q^3+9p^5q\Sigma^3+9p^4q^2\Sigma^3+27p^3\Sigma^6+\mathfrak{A} \end{aligned} \tag{3.13}$$

then we have:

$$\Lambda_1(\Sigma,p,q) = \sqrt[3]{\frac{\sqrt[3]{2}p^4q^2}{3\sqrt[3]{\mathfrak{B}}\Sigma^3}+\frac{\sqrt[3]{2}p^3}{\sqrt[3]{\mathfrak{B}}}+\frac{\sqrt[3]{2}p^2q}{\sqrt[3]{\mathfrak{B}}}+\frac{\sqrt[3]{\mathfrak{B}}}{3\sqrt[3]{2}\Sigma^3}+\frac{p^2q}{3\Sigma^3}+1} \tag{3.14}$$

$$\begin{aligned} \Lambda_2(\Sigma,p,q) &= \frac{1}{18\ 6^{2/3}\mathfrak{B}^{2/3}p\Sigma^5}\left[\frac{2^{2/3}\mathfrak{B}^{2/3}+2\sqrt[3]{\mathfrak{B}}\left(p^2q+3\Sigma^3\right)+2\sqrt[3]{2}p^2\left(p^2q^2+3p\Sigma^3+3q\Sigma^3\right)}{\sqrt[3]{\mathfrak{B}}\Sigma^3}\right]^{2/3}\times \\ &\Big[6\mathfrak{B}^{2/3}p^2\Sigma^3(p-q)-\sqrt[3]{2}\mathfrak{B}^{4/3}+2^{2/3}\mathfrak{B}\left(p^2q+3\Sigma^3\right)+ \\ &2\sqrt[3]{2\mathfrak{B}}p^2\left(p^4q^3+3p^3q\Sigma^3+6p^2q^2\Sigma^3+9p\Sigma^6+9q\Sigma^6\right) \\ &\quad -2\ 2^{2/3}p^4\left(p^2q^2+3p\Sigma^3+3q\Sigma^3\right)^2\Big] \end{aligned} \tag{3.15}$$

3.2 The issue of the Ricci-flat metric

One main question is whether the metric arising from the Kähler quotient, which is encoded in eq. (3.11) is Ricci-flat. A Ricci-flat metric on the crepant resolution of the singularity $\mathbb{C}^3/\mathbb{Z}_3$, namely on $\mathcal{O}_{\mathbb{P}^2}(-3)$, is known in explicit form from the work of Calabi[15] [42], yet it is not a priori obvious that the metric defined by the Kähler potential (3.11) is that one. The true answer is that it is not, as we show later on. Indeed we are able to construct directly the Kähler potential for the resolution of $\mathbb{C}^n/\mathbb{Z}_n$, for any $n \geq 2$, in particular determining the unique Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^2}(-3)$ with the same isometries as the metric (3.11) and comparing the two we see that they are different. Here we stress that the metric defined by (3.11) obviously depends on the level parameters ζ_1, ζ_2 while the Ricci-flat one is unique up to an overall scale factor. This is an additional reason to understand a priori that (3.11) cannot be the Ricci flat metric.

Actually Calabi in [42] found an easy form of the Kähler potential of a Ricci-flat metric on the canonical bundle of a Kähler-Einstein manifold, and that result applies to the cases of the canonical bundle of $\mathbb{P}^2$. However, in view of applications to cases where we shall consider the canonical bundles of manifolds which are not Kähler-Einstein, in the section we stick with our strategy of using the metric coming from the Kähler quotient as a starting point.

3.2.1 The Ricci-flat metric on $Y_{[3]} = \mathcal{O}_{\mathbb{P}^2}(-3)$

As we have noticed above the HKLR Kähler metric defined by the Kähler potential (3.11) depends only on the variable Σ defined in eq. (3.12). It follows that the HKLR Kähler metric admits U(3) as an isometry group, which is the hidden invariance of Σ. The already addressed question is whether the HKLR metric can be Ricci-flat. An almost immediate result is that a Ricci-flat Kähler metric depending only on the sum of the squared moduli of the complex coordinates is unique (up to a scale factor) and we can give a general formula for it.

We can present the result in the form of a theorem.

[15] Such metrics were also re-discovered in the physics literature in [52].

Theorem 3.1. *Let $\mathcal{M}_n$ be a non-compact n-dimensional Kähler manifold admitting a dense open coordinate patch z_i, $i = 1, \dots, n$ which we can identify with the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, the bundle structure being exposed by the coordinate transformation:*

$$z_i \,=\, u_i\, w^{\frac{1}{n}} \quad , \quad (i = 1, \dots, n-1) \qquad ; \quad z_n \,=\, w^{\frac{1}{n}} \tag{3.16}$$

where u_i is a set of inhomogenous coordinates for $\mathbb{P}^{n-1}$. The Kähler potential $\mathcal{K}_n$ of a U(n) *isometric Kähler metric on $\mathcal{M}_n$ must necessarily be a real function of the unique real variable $\Sigma = \sum_{i=1}^{n} |z_i|^2$. If we require that metric should be Ricci-flat, the Kähler potential is uniquely defined and it is the following one:*

$$\mathcal{K}_n(\Sigma) \,=\, k + \frac{(\Sigma^n + \ell^n)^{-\frac{n-1}{n}} \left((n-1)\,(\Sigma^n + \ell^n) - \ell^n\,(\Sigma^{-n}\ell^n + 1)^{\frac{n-1}{n}}\ {}_2F_1\left(\frac{n-1}{n}, \frac{n-1}{n}; \frac{2n-1}{n}; -\ell^n\Sigma^{-n}\right)\right)}{n-1} \tag{3.17}$$

where k is an irrelevant additive constant and $\ell > 0$ is a constant that can be reabsorbed by rescaling all the complex coordinates by a factor ℓ, namely $z_i \to \ell \widetilde{z}_i$.

Proof 3.1.1. The proof of the above statement is rather elementary. It suffices to recall that the Ricci tensor of any Kähler metric $\mathbf{g}_{ij^\star} \,=\, \partial_i \partial_{j^\star} \mathcal{K}(z, \overline{z})$ can always be calculated as follows:

$$\mathrm{Ric}_{ij^\star}[\mathbf{g}] \,=\, \partial_i\, \partial_{j^\star}\, \log\left[\mathrm{Det}\,[\mathbf{g}]\right] \tag{3.18}$$

In order for the Ricci tensor to be zero it is necessary that $\mathrm{Det}\,[\mathbf{g}]$ be the square modulus of a holomorphic function $|F(z)|^2$, on the other hand under the hypotheses of the theorem it is a real function of the real variable Σ. Hence it must be a constant. It follows that we have to impose the equation:

$$\mathrm{Det}\,[\mathbf{g}] \,=\, \ell^2 \,=\, \mathrm{const} \tag{3.19}$$

Let $\mathcal{K}(\Sigma)$ be the sought for Kähler potential, calculating the Kähler metric and its determinant we find:

$$\mathrm{Det}\,[\mathbf{g}] \,=\, \Sigma^{n-1}\mathcal{K}(\Sigma)' \left(\Sigma^2 \mathcal{K}(\Sigma)'' + \Sigma\, \mathcal{K}(\Sigma)'\right) \tag{3.20}$$

Inserting eq. (3.20) into eq. (3.19) we obtain a non linear differential equation for $\mathcal{K}(\Sigma)$ of which eq. (3.17) is the general integral. This proves the theorem. $\diamondsuit$

3.2.2 Particular cases

It is interesting to analyze particular cases of the general formula (3.17).

The case $n = 2$: Eguchi-Hanson. The case $n = 2$ yielding a Ricci flat metric on $\mathcal{O}_{\mathbb{P}^1}(-2)$ is the Eguchi-Hanson case namely the crepant resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$. This is known to be a HyperKähler manifold and all HyperKähler metrics are Ricci-flat. Hence also the HKLR metric must be Ricci-flat and identical with the one defined by eq. (3.17). Actually we find:

$$\begin{aligned}
\mathcal{K}_2(\Sigma) \;&=\; \left(\Sigma^2 + \ell^2\right)^{-\frac{1}{2}} \left(\left(\Sigma^2 + \ell^2\right) - \ell^2 \left(\Sigma^{-2}\ell^2 + 1\right)^{\frac{1}{2}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\ell^2\Sigma^{-2}\right)\right) \\
&=\; \sqrt{\Sigma^2 + \ell^2} - \ell \log\left(\sqrt{\Sigma^2 + \ell^2} + \ell\right) + \ell \log(\Sigma) \,+\, \mathrm{const}
\end{aligned} \tag{3.21}$$

which follows from the identification of the hypergeometric function with combinations of elementary transcendental functions occurring for special values of its indices. The second transcription of the function is precisely the Kähler potential of the Eguchi-Hanson metric in its HKLR-form as it arises from the Kronheimer construction (see for instance [6]).

THE CASE $n = 3$: $\mathcal{O}_{\mathbb{P}^2}(-3)$. The next case is that of interest for the D3-brane solution. For $n = 3$, setting $\ell = 1$, which we can always do by a rescaling of the coordinates, we find:

$$
\begin{aligned}
\mathcal{K}_{Rflat}(\Sigma) &= \frac{2\left(\Sigma^3+1\right) - \left(\frac{1}{\Sigma^3}+1\right)^{2/3} {}_2F_1\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};-\frac{1}{\Sigma^3}\right)}{2\left(\Sigma^3+1\right)^{2/3}} \\
&= \frac{2(\Sigma^3+1) - {}_2F_1\left(\frac{2}{3},1;\frac{5}{3};\frac{1}{\Sigma^3+1}\right)}{2\left(\Sigma^3+1\right)^{2/3}}
\end{aligned}
\tag{3.22}
$$

The second way of writing the Kähler potential follows from one of the standard Kummer relations among hypergeometric functions. There is a third transcription that also in this case allows to write it in terms of elementary transcendental functions. Before considering it we use eq. (3.22) to study the asymptotic behavior of the Kähler potential for large values of Σ. We obtain;

$$
\mathcal{K}_{Rflat}(\Sigma) \overset{\Sigma\to\infty}{\approx} \Sigma - \frac{1}{6\Sigma^2} + \frac{1}{45\Sigma^5} + \mathcal{O}\left(\frac{1}{\Sigma^7}\right) \tag{3.23}
$$

Eq.(3.23) shows that the Ricci-flat metric is asymptotically flat since the Kähler potential approaches that of $\mathbb{C}^3$.

As anticipated, there is an alternative way of writing the Kähler potential (3.22) which is the following:

$$
\begin{aligned}
\mathcal{K}_{Rflat}(\Sigma) = \; & \frac{\pi}{2\sqrt{3}} + \frac{1}{6}\Bigg(6\sqrt[3]{\Sigma^3+1} + 2\log\left(\sqrt[3]{\Sigma^3+1}-1\right) \\
& - \log\left(\left(\Sigma^3+1\right)^{2/3} + \sqrt[3]{\Sigma^3+1}+1\right) - 2\sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{\Sigma^3+1}+1}{\sqrt{3}}\right)\Bigg)
\end{aligned}
\tag{3.24}
$$

The identity of eq. (3.24) with eq. (3.22) can be worked with analytic manipulations that we omit. The representation (3.24) is particularly useful to explore the behavior of the Kähler potential at small values of Σ. We immediately find that:

$$
\mathcal{K}_{Rflat}(\Sigma) \overset{\Sigma\to 0}{\approx} \log[\Sigma] + \frac{\pi}{2\sqrt{3}} + \mathcal{O}\left(\Sigma^6\right) \tag{3.25}
$$

The behavior of $\mathcal{K}_{Rflat}(\Sigma)$ is displayed in fig.3.

3.3 The harmonic function in the case $Y_{[3]} = \mathcal{O}_{\mathbb{P}^2}(-3)$

Let us now consider the equation for a harmonic function $H(z,\overline{z})$ on the background of the Ricci-flat metric of $Y_{[3]}$ that we have derived in the previous sections. Once again we suppose that $H = H(\Sigma)$ is a function only of the real variable Σ, *viz.* $R = \sqrt{\Sigma}$. For the Ricci-flat metric the Laplacian equation takes the simplified form: $\partial_i\left(g^{ij^\star}\,\partial_{j^\star}H(\Sigma)\right) = 0$, since the determinant of the metric is constant. Using the Kähler metric that follows from the Kähler potential $K_{Rflat}(\Sigma)$ defined by eq.s (3.22),(3.24), we obtain a differential equation that upon the change of variable $\Sigma = \sqrt[3]{r}$ takes the following form:

$$
3r(r+1)C''(r) + (5r+3)C'(r) \;=\; 0 \tag{3.26}
$$

The general integral eq. (3.26) is displayed below:

$$
C(r) \;=\; \kappa + \lambda\left(\log\left(1-\sqrt[3]{r+1}\right) - \frac{1}{2}\log\left((r+1)^{2/3} + \sqrt[3]{r+1}+1\right) - \sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{r+1}+1}{\sqrt{3}}\right)\right) \tag{3.27}
$$

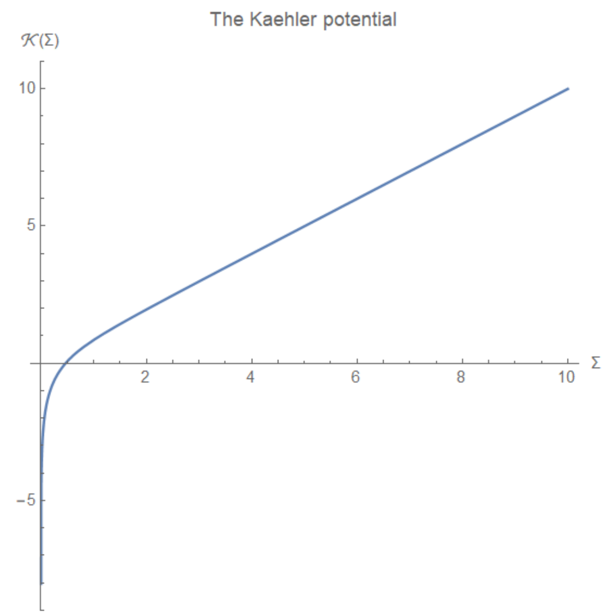

Figure 3: The plot of the Kähler potential $\mathcal{K}_{Rflat}(\Sigma)$ for the Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^2}(-3)$. The asymptotic flatness of the metric is evident from the plot. For large values of Σ it becomes a straight line with angular coefficient 1.

κ, λ being the two integration constants. We fix these latter with boundary conditions. We argue in the following way: if the transverse space to the brane were the original $\mathbb{C}^3/\mathbb{Z}_3$ instead of the resolved variety $\mathcal{O}_{\mathbb{P}^2}(-3)$, then the harmonic function describing the D3-brane solution would be the following:

$$H_{orb}(R) = 1 + \frac{1}{R^4} \quad ; \quad R \equiv \sqrt{\Sigma} = \sqrt{\sum_i^2 |z_i|^2} = \sqrt[6]{r} \tag{3.28}$$

The asymptotic identification for $R \to \infty$ of the Minkowski metric in ten dimension would be guaranteed, while at small values of R we would find (via dimensional transmutation) the standard AdS$_5$-metric times that of $\mathbb{S}^5$ (see the following eq.s (3.33) and (3.34)). In view of this, naming R the square root of the variable Σ, we fix the coefficients κ, λ in the harmonic function $H_{res}(R)$ in such a way that for large values of R it approaches the harmonic function pertaining to the orbifold case (3.28). The asymptotic expansion of the function: $H_{res}(R) \equiv C(r^6)$ is the following one:

$$H_{res}(R) \overset{R\to\infty}{\approx} \left(\lambda - \frac{\pi\kappa}{2\sqrt{3}}\right) - \frac{1}{2}\kappa\left(\frac{1}{R}\right)^4 + O\left(\left(\frac{1}{R}\right)^5\right) \tag{3.29}$$

Hence the function $H_{res}(R)$ approximates the function $H_{orb}(R)$ if we set $\kappa = 2$, $\lambda = \frac{\pi}{\sqrt{3}}$. In this way we conclude that:

$$\begin{aligned} H_{res}(R) = & \frac{1}{3}\left(2\log\left(\sqrt[3]{R^6+1}-1\right) - \log\left(\left(R^6+1\right)^{2/3} + \sqrt[3]{R^6+1}+1\right)\right. \\ & \left. -2\sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{R^6+1}+1}{\sqrt{3}}\right)\right) + \frac{\pi}{\sqrt{3}} \end{aligned} \tag{3.30}$$

The overall behavior of the function $H_{res}(R)$ is displayed in fig.4.

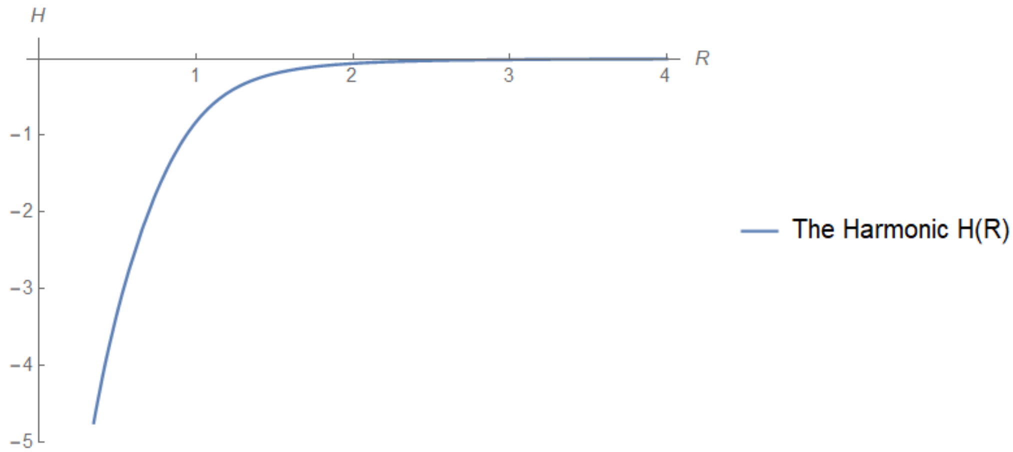

Figure 4: The plot of the harmonic function $H_{res}(R)$ for the Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^2}(-3)$.

3.4 The asymptotic limits of the Ricci-flat metric for the D3-brane solution on $\mathcal{O}_{\mathbb{P}^2}(-3)$

In the case of a standard D3-brane on $Y_{[3]} = \mathbb{C}^3 \simeq \mathbb{R}^6$ one writes the same ansatz as in eq. (2.2) and (2.26-2.27) where now the Kähler metric is $\mathbf{g}_{\alpha\beta^\star} = \delta_{\alpha\beta^\star}$ Rewriting the complex coordinates in terms of polar coordinates $z_1 = e^{\mathrm{i}\varphi_1} R\cos\phi$, $z_2 = e^{\mathrm{i}\varphi_2} R\cos\chi\sin\phi$, $z_3 = e^{\mathrm{i}\varphi_3} R\sin\chi\sin\phi$ we obtain that:

$$\mathrm{ds}^2_{\mathbb{C}^3} \equiv \sum_{i=1}^{3} |dz_i|^2 = dR^2 + R^2\,\mathrm{ds}^2_{\mathbb{S}^5} \tag{3.31}$$

where:

$$\mathrm{ds}^2_{\mathbb{S}^5} = d\varphi_1{}^2\cos^2\phi + \sin^2\phi\left(d\varphi_2{}^2\cos^2\chi + d\varphi_3{}^2\sin^2\chi + d\chi^2\right) + d\phi^2 \tag{3.32}$$

is the SO(6)-invariant metric of a 5-sphere in polar coordinates. In other words the Ricci-flat Kähler metric $\mathrm{ds}^2_{\mathbb{C}^3}$ (which is also Riemann-flat) is that of the *metric cone* on the Sasaki-Einstein metric of $\mathbb{S}^5$. At the same time the SO(6)-invariant harmonic function on $\mathbb{C}^3$ is given by the already quoted $H_{orb}(R)$ in (3.28), and the complete 10-dimensional metric of the D3-brane solution takes the form:

$$\mathrm{ds}^2_{10|orb} = \frac{1}{\sqrt{1+\frac{1}{R^4}}}\,\mathrm{ds}^2_{\mathrm{Mink}_{1,3}} + \sqrt{1+\frac{1}{R^4}}\left(dR^2 + R^2\,\mathrm{ds}^2_{\mathbb{S}^5}\right) \tag{3.33}$$

For $R \to \infty$ the metric (3.33) approaches the flat Minkowski metric in $d = 10$, while for $R \to 0$ it approaches the following metric:

$$\mathrm{ds}^2_{10|orb} \overset{R\to 0}{\approx} \underbrace{R^2\,\mathrm{ds}^2_{\mathrm{Mink}_{1,3}} + \frac{dR^2}{R^2}}_{\mathrm{AdS}_5} + \underbrace{\mathrm{ds}^2_{\mathbb{S}^5}}_{\mathbb{S}^5} \tag{3.34}$$

Let us now cosider the asymptotic behavior of the Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^2}(-3)$. In order to obtain a precise comparison with the flat orbifold case the main technical point is provided by the transcription of the $\mathbb{S}^5$-metric in terms of coordinates well adapted to the Hopf fibration:

$$\mathbb{S}^5 \xrightarrow{\pi} \mathbb{P}^2 \quad ; \quad \forall\, p \in \mathbb{P}^2 \quad \pi^{-1}(p) \sim \mathbb{S}^1 \tag{3.35}$$

To this effect let $Y = \{u, v\}$ be a pair of complex coordinates for $\mathbb{P}^2$ such that the standard Fubini-Study metric on this compact 2-fold is given by:

$$\mathrm{ds}^2_{\mathbb{P}^2} = g^{\mathbb{P}^2}_{ij^\star}\, dY^i\, d\overline{Y}^{j^\star} \equiv \frac{dY\cdot d\overline{Y}}{1+Y\cdot\overline{Y}} - \frac{\left(\overline{Y}\cdot dY\right)\left(Y\cdot d\overline{Y}\right)}{\left(1+Y\cdot\overline{Y}\right)^2} \tag{3.36}$$

the corresponding Kähler 2-form being $\mathbb{K}_{\mathbb{P}^2} = \frac{\mathrm{i}}{2\pi}\, g^{\mathbb{P}^2}_{ij^\star}\, dY^i \wedge d\overline{Y}^{j^\star}$. Introducing the one form: $\Omega = \frac{\mathrm{i}\left(Y\cdot d\overline{Y} - \overline{Y}\cdot dY\right)}{2\left(1+Y\cdot\overline{Y}\right)}$ whose exterior derivative is the Kähler 2-form, $d\Omega = 2\pi\,\mathbb{K}_{\mathbb{P}^2}$, the metric of the five-sphere in terms of these variables is the following one:

$$\mathrm{ds}^2_{\mathbb{S}^5} = \mathrm{ds}^2_{\mathbb{P}^2} + (d\varphi + \Omega)^2 \tag{3.37}$$

where the range of the coordinate φ spanning the $\mathbb{S}^1$ fiber is $\varphi \in [0, 2\pi]$. In this way the flat metric on the metric cone on $\mathbb{S}^5$, namely (3.31) can be rewritten as follows:

$$\mathrm{ds}^2_{\mathbb{C}^3} = dR^2 + R^2\,\mathrm{ds}^2_{\mathbb{P}^2} + R^2\,(d\varphi + \Omega)^2 \tag{3.38}$$

3.4.1 Comparison of the Ricci-flat metric with the orbifold metric

In order to compare the exact Ricci-flat metric streaming from the Kähler potential (3.22) with the metric (3.12) it suffices to turn to toric coordinates

$$z_1 = u\sqrt[3]{w}\,, \quad z_2 = v\sqrt[3]{w}\,, \quad z_3 = \sqrt[3]{w} \quad ; \quad \Sigma = (1+\varpi)\,\mathfrak{f}^{1/3} \quad ; \quad \varpi = |u|^2 + |v|^2 \quad ; \quad \mathfrak{f} = |w|^2 \tag{3.39}$$

The toric coordinates $\{u, v\} \equiv Y$ span the exceptional divisor $\mathbb{P}^2$ while w is the fiber coordinate in the bundle. Setting:

$$w = e^{\mathrm{i}\psi}|w| = e^{\mathrm{i}\psi}\left(\frac{R^2}{1+|u|^2+|v|^2}\right)^{\frac{3}{2}} \tag{3.40}$$

we obtain:

$$\begin{aligned} \mathrm{ds}^2_{Rflat} &= h(R)dR^2 + f(R)\,\mathrm{ds}^2_{\mathbb{P}^2} + g(R)\,(d\psi + 3\,\Omega)^2 \\ f(R) &= \sqrt[3]{R^6+1} \quad ; \quad h(R) = \frac{R^4}{(R^6+1)^{2/3}} \quad ; \quad g(R) = \frac{R^6}{9\,(R^6+1)^{2/3}} \end{aligned} \tag{3.41}$$

From eq. (3.41) we derive the asymptotic form of the metric for large values of R, namely:

$$\mathrm{ds}^2_{Rflat} \overset{R\to\infty}{\approx} dR^2 + R^2\mathrm{ds}^2_{\mathbb{P}_2} + R^2\left(\frac{\mathrm{d}\psi}{3} + \Omega\right)^2 \tag{3.42}$$

The only difference between eq. (3.38) and eq. (3.42) is the range of the angular value $\varphi = \frac{\psi}{3}$. Because of the original definition of the angle ψ, the new angle $\varphi \in \left[0, \frac{2\pi}{3}\right]$ takes one third of the values. This means that the asymptotic metric cone is quotiened by $\mathbb{Z}_3$ as it is natural since we resolved the singularity $\mathbb{C}^3/\mathbb{Z}_3$.

3.4.2 Reduction to the exceptional divisor

The other important limit of the Ricci-flat metric is its reduction to the exceptional divisor $\mathcal{ED}$. In the present case the only fixed point for the action of $\Gamma = \mathbb{Z}_3$ on $\mathbb{C}^3$ is provided by the origin $z_{1,2,3} = 0$ which, comparing with eq. (3.39), means $w = 0 \Rightarrow \mathfrak{f} = 0$. This is the equation of the exceptional divisor which is created by the blowup of the unique singular point. In the basis of the complex toric coordinates $Y^i \equiv \{u, v, w\}$, the Kähler metric derived from the Kähler potential (3.22) has the following appearance:

$$g^{Rflat}_{ij^\star} = \begin{pmatrix} \frac{v\overline{v}+\mathfrak{f}(\varpi+1)^4+1}{(\varpi+1)^2(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & -\frac{v\overline{u}}{(\varpi+1)^2(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & \frac{w(\varpi+1)^2\overline{u}}{3(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} \\ -\frac{u\overline{v}}{(\varpi+1)^2(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & \frac{u\overline{u}+\mathfrak{f}(\varpi+1)^4+1}{(\varpi+1)^2(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & \frac{w(\varpi+1)^2\overline{v}}{3(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} \\ \frac{u(\varpi+1)^2\overline{w}}{3(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & \frac{v(\varpi+1)^2\overline{w}}{3(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} & \frac{(\varpi+1)^3}{9(\mathfrak{f}(\varpi+1)^3+1)^{2/3}} \end{pmatrix} \tag{3.43}$$

where the invariants $\mathfrak{f}, \varpi$ are defined in equation (3.39). Hence the reduction of the metric to the exceptional divisor is obtained by setting $\mathrm{d}w = \mathrm{d}\overline{w} = 0$ in the line element $\mathrm{ds}^2_{Rflat} \equiv g^{Rflat}_{ij^\star} dY^i \, d\overline{Y}^{j^\star}$ and performing the limit $\mathfrak{f} \to 0$ on the result. We obtain:

$$\mathrm{ds}^2_{Rflat} \xrightarrow{\mathcal{ED}} \mathrm{ds}^2_{\mathbb{P}^2} \equiv \frac{\mathrm{d}v\,(\mathrm{d}\overline{v} + u\,\overline{u}\mathrm{d}\overline{v} - u\overline{v}\,\mathrm{d}\overline{u}) + \mathrm{d}u\,(\mathrm{d}\overline{u} + v\,\overline{v}\mathrm{d}\overline{u} - \overline{u}\,v\,\mathrm{d}\overline{v})}{(1 + u\overline{u} + v\overline{v})^2} \tag{3.44}$$

which is the standard Fubini-Study metric on $\mathbb{P}^2$ obtained from the Kähler potential:

$$\mathcal{K}^{FS}_{\mathbb{P}^2}(\varpi) \,=\, \log\left(1 + \varpi\right) \tag{3.45}$$

As we see, the metric on the exceptional divisor obtained from the Ricci-flat metric has no memory of the Fayet Iliopoulos (or stability parameters) p, q which characterize instead the HKLR metric obtained from the Kronheimer construction. This is obvious since the Ricci-flat metric does not depend on p, q. On the other hand the HKLR metric, that follows from the Kähler potential (3.11), strongly depends on the Fayet Iliopoulos parameters $\zeta_1 = p\,,\, \zeta_2 = q$ and one naturally expects that the reduction of ds^2_{HKLR} to the exceptional divisor will inherit such a dependence. Actually this is not the case since the entire dependence from p, q of the HKLR Kähler potential, once reduced to $\mathcal{ED}$, is localized in an overall multiplicative constant and in an irrelevant additive constant. This matter of fact is conceptually very important in view of our conjecture that the Ricci-flat metric is completely determined, by means of the Monge-Ampère equation, from the Kähler metric on the exceptional divisor, as it is determined by the Kronheimer construction. In the present case where, up to a multiplicative constant, *i.e. a homothety* there is only one Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^2}(-3)$ with the prescribed isometries, our conjecture might be true only if the reduction of the HKLR metric to the exceptional divisor is unique and p, q-independent, apart from overall rescalings. It is very much reassuring that this is precisely what actually happens.

4 The case $Y \to \mathbb{C}^3/\mathbb{Z}_4$ and the general problem of determining a Ricci-flat metric

The next case of interest to us at present is the resolution $Y \to \mathbb{C}^3/\mathbb{Z}_4$ whose associated Kronheimer construction was studied in detail in [7]. (A study of $\mathbb{C}^3/\mathbb{Z}_4$ as a non-complete intersection affine variety in $\mathbb{C}^9$ is presented in the Appendix.) The corresponding MacKay quiver is displayed in fig.5. Differently from the case of the resolution $Y \to \mathbb{C}^3/\mathbb{Z}_3$ studied in section 3, here the HKLR Kähler metric cannot be derived explicitly since the moment map equations form a system of algebraic equations of higher degree. Yet as it was explained in [7] one can work out the restriction of such metric to the compact component of the exceptional divisor which is the second Hirzebruch surface $\mathbb{F}_2$. Indeed it was shown that the quotient singularity $\mathbb{C}^3/\mathbb{Z}_4$ can be completely resolved by tot$K_{\mathbb{F}_2}$ [7], that denotes the total space of the canonical bundle over the second Hirzebruch surface.

Hence the main goal we would like to achieve is the construction of a Ricci-flat Kähler metric on tot$K_{\mathbb{F}_2}$ which restricted to the base $\mathbb{F}_2$ of the bundle hopefully coincides with Kähler metric on the same surface provided by the Kronheimer construction.

Being a non-compact Calabi-Yau variety the existence of a Ricci-flat Kähler metric on tot$K_{\mathbb{F}_2}$ is not implied by the classic Yau theorem, valid for smooth compact manifolds. To ask whether Ricci-flat metrics do exist, one has to specify boundary conditions. We will be interested in metrics that, just as in the previous example, are asymptotically conical, namely of the form[16]

$$\mathrm{ds}^2(Y) \approx dR^2 + R^2\mathrm{ds}^2(X_5) \tag{4.1}$$

[16] Note that without specifying the boundary conditions there can exist more than one Ricci-flat metric. Explicit examples of non-asymptotically conical Ricci-flat metrics in six real dimensions can be found in [43].

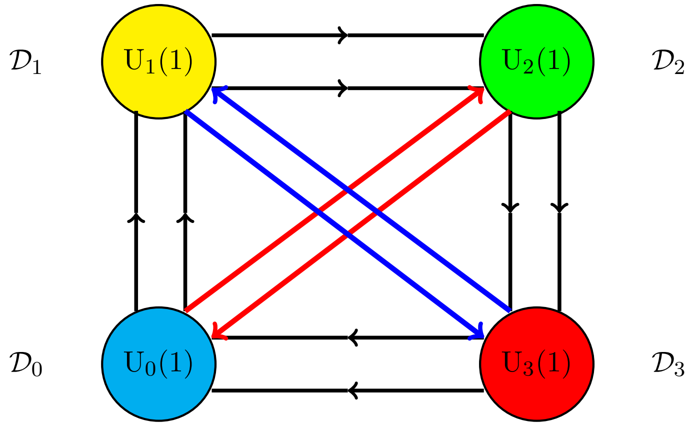

Figure 5: The quiver diagram describing the $\mathbb{C}^3/\mathbb{Z}_4$ singular quotient and codifying its resolution via Kähler quotient à la Kronheimer. The same quiver diagram codifies the construction of the corresponding gauge theory for a stack of D3-branes. Each node is associated with one of the 4 irreducible representations of $\mathbb{Z}_4$ and in each node we located one of the $\mathrm{U}_\mathrm{i}(1)$ groups with respect to which we perform the Kähler quotient. This is the case of one D3-brane. For N D3–branes, all gauge groups $\mathrm{U}_\mathrm{i}(1)$ are promoted to $\mathrm{U}_\mathrm{i}(\mathrm{N})$.

for a suitable radial coordinate approaching $R \to \infty$. Essentially by definition, $\mathrm{ds}^2(X_5)$ is a Sasaki-Einstein metric on a compact manifold (or orbifold) X_5. Then we fix the boundary conditions for our metric by requiring that asymptotically it approaches the cone over $\mathbb{S}^5/\mathbb{Z}_4$. With this boundary condition[17] the theorems in [28] imply the existence of a unique Ricci-flat Kähler metric in every Kähler class of the resolved variety Y. Analogous existence results for isolated quotient singularities $\mathbb{C}^m/\Gamma$ were given in [44] and later extended in [45] and [46] for crepant resolutions of general isolated conical singularities. See also [47] for of applications of the general existence results in the toric context, including the resolution of the conical singularities on the $Y^{p,q}$ Sasaki-Einstein five-manifolds [48].

The existence results are analogous to Yau's theorem in the compact case. In fact, recently there has been some renewed interest and activity in this area, with some new results concerning for example the existence of Sasaki-Einstein manifolds, outside the toric realm. These results are related to the idea of "stability". For reference, recent work on this subject include [49, 50].

For many purposes, knowledge of the existence of a metric, together with some of its key properties, can be sufficient for extracting interesting physical information. This is true also in the case of the AdS/CFT correspondence. However, if one is interested in constructing the metrics explicitly, namely write them down in some coordinate systems, then the existence theorems are not helpful, because they are not constructive (as far as we know).

The classic examples of explicit Ricci-flat Kähler metrics in real dimension four include Eguchi-Hanson, Gibbons-Hawking, Taub-NUT, Atiyah-Hitchin. In real dimension six, for a long time the resolved and deformed metrics on the conifold singularity constructed by Candelas and de la Ossa [51] were the only (non trivial) known examples of explicit Ricci-flat Kähler metrics. The so-called "resolved conifold" metric is a metric on the total space of the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, the isometry group is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ and asymptotically it approaches the cone over the Sasaki-Einstein manifold $\mathrm{T}^{1,1}$ (with the same isometry). In other cases, different kind of resolutions exist, where instead of a $\mathbb{P}^1$ one replaces the singularity with a compact four-dimensional manifold (or orbifold) $\mathcal{M}_4$. A general ansatz that yields explicit Ricci-flat Kähler metrics was constructed by Page and Pope (in any dimension) [52], but this is somewhat limited as it *assumes* that the metric induced on $\mathcal{M}_4$ is Kähler-Einstein.

[17] The results in [28] require some more precise estimate on the fall-off of the metric at infinity.

Explicit Kähler-Einstein metrics on smooth four-dimensional manifolds are known only for $\mathcal{M}_4 = \mathbb{P}^2$ and $\mathcal{M}_4 = \mathbb{P}^1 \times \mathbb{P}^1$. The former leads to the construction of an explicit Ricci-flat Kähler metric on the total space of $\mathcal{O}_{\mathbb{P}^2}(-3) \simeq \mathrm{tot}K_{\mathbb{P}^2}$, which is the resolution of the quotient singularity $\mathbb{C}^3/\mathbb{Z}_3$ and was fully described in section 3 (see also[42]). The latter leads to the construction of an explicit Ricci-flat Kähler metric on the total space of $\mathrm{tot}K_{\mathbb{P}^1\times\mathbb{P}^1}$, which is the resolution of the conical singularity (conifold)/$\mathbb{Z}_2$. The corresponding Sasaki-Einstein manifolds at infinity are, respectively, $\mathbb{S}^5/\mathbb{Z}_3$ and $\mathrm{T}^{1,1}/\mathbb{Z}_2$. For the case of $\mathrm{tot}K_{\mathbb{P}^1\times\mathbb{P}^1}$, a generalisation was constructed [53], namely an explicit Ricci-flat Kähler metric that depends on the two independent Kähler classes parameters: this construction however uses the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry and as a result the metric is co-homogeneity one, although it does not fit in the ansatz of [42] and [52]. Recently, the ansatz of [42, 52] was used to produce explicit Ricci-flat Kähler metrics on the canonical bundle of generalised flag manifolds [54]. Extensions that include the dependence on several Kähler class parameters have appeared in [55, 56].

4.1 The Ricci-flat Kähler metric on $\mathrm{tot}K_{\mathbb{F}_1}$

The metric that we shall present in the sequel has some distinctive features that are shared with an explicit Ricci-flat Kähler metric on $\mathrm{tot}K_{\mathbb{F}_1}$, where $\mathbb{F}_1$ is the first Hirzebruch surface, *i.e.*, the first del Pezzo surface dP_1, constructed in [57]. This metric is many ways "more complicated" than all the other metrics mentioned above. Let us summarise some of its salient properties:

1. Asymptotically it approaches the cone over the Sasaki-Einstein manifold[18] $Y^{2,1}$.
2. The isometry group is $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$.
3. It is cohomogeneity two. In particular, there is a homogeneous base, given by a round $\mathbb{P}^1$, and then the metric depends non-trivially on two coordinates.
4. It is *toric*, in that there is a $\mathrm{U}(1)^3 \in \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ subgroup of isometries that leaves invariant the Kähler form, and contains the torus of the toric three-fold $\mathrm{tot}K_{\mathbb{F}_1}$. This group allows one to introduce three moment map coordinates and three angular coordinates ("action-angle" coordinates system).
5. It also possesses an additional "hidden symmetry" corresponding to the existence of a so-called Hamiltonian two-form [14], that implies the existence of a coordinate system (called "orthotoric") in which the metric components are all given in terms of functions of one variable.
6. Imposing this extra symmetry however, comes at the price of loosing one of the two Kähler class parameters. Indeed it was later demonstrated in [29] that the two-parameter metric (that is known to exist thanks to the general theorems of [45, 46]) does not posses such Hamiltonian two-form.
7. The metric induced on exceptional divisor $\mathcal{M}_4 = \mathbb{F}_1$ is obviously Kähler, but it is not Einstein. Indeed, a Kähler-Einstein metric on $\mathbb{F}_1$ does not exist.
8. In [14] (further explored in detail in [59]) it was shown that this metric is part of a family of (in general only *partial*[19]) resolutions of the conical Ricci-flat metrics on the whole family of $Y^{p,q}$ Sasaki-Einstein manifolds.

[18] Incidentally, $Y^{2,1}$ can also be viewed as circle bundle over $\mathbb{F}_1$. See section 5 of [58].

[19] This means that for general p and q the compact divisor $\mathcal{M}_4$ has orbifold singularities [59, 14]. This is because the metric ansatz is "too simple" to account for all the necessary Kähler class parameters; but completely resolved metrics are known to exist [47]. For the special case $p = 2$ and $q = 1$ the metric is completely smooth. We also note that in [14] were constructed different types of partial resolutions, corresponding to various "chambers". Moreover, the paper discusses the case of general dimension, while for our purposes we shall focus on the case of real dimension $d = 6$.

9. In [29] it is given a relation between the orthotoric coordinates and a set of complex coordinates that is well adapted to the complex structure of tot$K_{\mathbb{F}_1}$, with one complex coordinate on the non-compact fiber $\mathbb{C}$, one coordinate on the fiber $\mathbb{P}^1$ in $\mathbb{F}_1$ and one coordinate on the base $\mathbb{P}^1$ in $\mathbb{F}_1$.

10. A set of local complex coordinates explicitly related to the orthotoric coordinates was given in section 2.2 of [14]. It would be interesting to work out the relation between these and the complex coordinates defined in [29].

Since, similarly to $\mathbb{F}_1$, also $\mathbb{F}_2$ does not admit a Kähler-Einstein metric, the Ricci-flat metric on tot$K_{\mathbb{F}_2}$ cannot be found through the Calabi ansatz [42, 52]. We expect the Ricci-flat metric on tot$K_{\mathbb{F}_2}$ to share many features with that on tot$K_{\mathbb{F}_1}$, summarised above. One difference is that at infinity it must approach the cone over the Sasaki-Einstein orbifold $\mathbb{S}^5/\mathbb{Z}_4$, as opposed to the cone over the Sasaki-Einstein manifold $Y^{2,1}$. The Ricci-flat metric on tot$K_{\mathbb{F}_2}$ will also be toric and moreover it should have again isometry group $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$. This immediately implies that the metric should be co-homogeneity two and in practice it leads to PDE's in two variables. For example, one can write the Monge-Ampere equation for the Kähler potential as a PDE in two variables, or similarly the corresponding equation for the symplectic potential. Without further assumptions, these equations are unlikely to be solvable in closed form.

A natural assumption to make is that the metric admits a Hamiltonian two-form, namely that it can be put in the orthotoric form. This is natural because the partial resolution of all the $Y^{p,q}$ singularities arise in this ansatz, with $p = 2$, $q = 1$ giving the complete resolution above. Strictly speaking the $p > q > 0$ should hold, however, it is known that by performing a scaling limit of the $Y^{p,q}$ Sasaki-Einstein metrics, one can recover the limiting cases $Y^{p,p} = \mathbb{S}^5/\mathbb{Z}_{2p}$ and $Y^{p,0} = \mathrm{T}^{1,1}/\mathbb{Z}_p$, suggesting that the partial resolution metrics may also be extended to these regimes of parameters[20].

5 A general set up for a metric ansatz with separation of variables

In the sequel we begin by considering a metric on a 6-dimensional manifold $\mathcal{M}_6$ which is Kähler and by construction admits $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ as an isometry group. This metric depends on two functions $\Upsilon(s)$ and $P(t)$ of two real coordinates s, t invariant with respect to the isometry group. The other coordinates are four angles, with ranges and periodicities specified according with the following summary table:

$$s \leq -3\,, \quad -\tfrac{3}{2} \leq t \leq 0, \quad 0 \leq \theta \leq \pi, \quad \phi \in [0, 2\pi]\,, \quad \tau \in [0, 2\pi]\,, \quad \chi \in [0, \tfrac{3}{2}\pi] \tag{5.1}$$

The metric, which is defined by means of the following vielbein

$$\begin{aligned}
E^1 &= \frac{1}{2}\sqrt{st}d\theta \\
E^2 &= \frac{1}{2}\sqrt{st}\sin\theta d\phi \\
E^3 &= \frac{1}{2}\sqrt{\frac{s-t}{3+s}}\,\Upsilon(s)ds \\
E^4 &= \frac{1}{2}\sqrt{t-s}P(t)\,dt \\
E^5 &= -\frac{1}{\sqrt{\frac{s-t}{3+s}}\,\Upsilon(s)}\left[-\frac{1}{2}t\left(d\tau + (1-\cos\theta)d\phi - \frac{2d\chi}{3}\right) + d\chi\right] \\
E^6 &= -\frac{1}{\sqrt{t-s}P(t)}\left[-\frac{1}{2}s\left(d\tau + (1-\cos\theta)d\phi - \frac{2d\chi}{3}\right) + d\chi\right]
\end{aligned} \tag{5.2}$$

[20] In fact, in Appendix A of [14] the metric ansatz of [52] is recovered in a limit.

is derived, by generalization, from the *orthotoric metrics* discussed in[21] [59, 14] where the relation of latter with the metrics on Sasakian 5-manifolds $Y^{p,q}$ is also presented. Although in those references it was assumed that $p > q$, presently we will consider setting $p = q = 2$ and show that this yields an orthotoric metric that we shall identify as a Ricci-flat Kähler metric on $\mathrm{tot}K_{\mathbb{W}P[112]}$. The asymptotic metric corresponds to a cone over the limiting case $Y^{2,2} = \mathbb{S}^5/\mathbb{Z}_4$ of the Sasaki-Einstein manifolds $Y^{p,q}$ [48].

The line-element:

$$\begin{aligned} \mathrm{ds}^2_{\mathrm{ort}} &= \sum_{i=1}^{6} \left(E^i\right)^2 &(5.3)\\ &= \frac{1}{4} s\, t\, \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{(s-t)\Upsilon(s)^2}{4(3+s)}\, ds^2 + \frac{1}{4}(t-s)P(t)^2\, \mathrm{d}t^2 \\ &\quad + \frac{(3+s)}{(s-t)\Upsilon(s)^2}\left[-\frac{1}{2} t \left(\mathrm{d}\tau + (1-\cos\theta)d\phi - \frac{2\mathrm{d}\chi}{3}\right) + d\chi\right]^2 \\ &\quad + \frac{1}{(t-s)P(t)^2}\left[-\frac{1}{2} s \left(\mathrm{d}\tau + (1-\cos\theta)d\phi - \frac{2\mathrm{d}\chi}{3}\right) + d\chi\right]^2 &(5.4) \end{aligned}$$

is Kählerian by construction since it admits the following closed Kähler 2-form:

$$\begin{aligned} \mathbb{K}_{\mathrm{ort}} &= E_1 \wedge E_2 + E_3 \wedge E_5 + E_4 \wedge E_6 \\ &= \frac{1}{2}\left\{ t \left[-\frac{1}{2}\cos\theta \mathrm{ds} \wedge d\phi + \frac{1}{2}\left(ds \wedge d\tau - \frac{2}{3} ds \wedge d\chi\right) + \frac{1}{2} ds \wedge d\phi\right] - ds \wedge d\chi \right\} \\ &\quad + \frac{1}{2}\left\{ s \left[-\frac{1}{2}\cos\theta \mathrm{dt} \wedge d\phi + \frac{1}{2}\left(\mathrm{dt} \wedge d\tau - \frac{2}{3} \mathrm{dt} \wedge d\chi\right) + \frac{1}{2} \mathrm{dt} \wedge d\phi\right] - \mathrm{dt} \wedge d\chi \right\} \\ &\quad + \frac{1}{4} st \sin\theta d\theta \wedge d\phi &(5.5) \end{aligned}$$

Indeed $\mathbb{K}_{\mathrm{ort}}$ is closed by construction and it is a Kähler 2-form since we have:

$$\mathbb{K}_{\mathrm{ort}} = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} \mathbb{J}_{\mathrm{ij}} E^i \wedge E^j \tag{5.6}$$

where:

$$\mathbb{J} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad ; \quad \mathbb{J}^2 = -\mathbf{1} \tag{5.7}$$

is an antisymmetric tensor which squares to minus the identity, namely it is a frame-index complex structure tensor.

It should be noted that the Kähler form in eq. (5.5) is independent from the two functions $\Upsilon(s)$ and $P(t)$, namely it is universal for an entire class of metrics.

[21] In particular, see the line element (4.1) in [59], after correcting some typos in that expression. The relation to our coordinates is given by $t = y - 1$, $s = x - 1$. Moreover, we have $\theta_{\mathrm{here}} = \theta_{\mathrm{there}}$, $\phi_{\mathrm{here}} = \phi_{\mathrm{there}}$, as well as $\chi_{\mathrm{here}} = \tau_{\mathrm{there}}$, $\tau_{\mathrm{here}} = 2\psi_{\mathrm{there}} + \frac{2}{3}\tau_{\mathrm{there}}$.

5.1 The orthotoric metric on $K_{\mathbb{W}P[112]}$

Within the general scope of the above described setup we have that the metric (5.4) is Ricci-flat for the following choice of the two functions parameterizing the line-element:

$$\Upsilon(s) = \sqrt{\frac{-s}{\frac{2}{3}s^2 - s + 3}} \quad ; \quad P(t) = \frac{1}{\sqrt{-\frac{2}{3}t^2 - t}} \tag{5.8}$$

With the choice (5.8), from eq. (5.4) we obtain:

$$\begin{aligned} \mathrm{ds}^2_{\mathrm{tot}K_{\mathbb{W}P[112]}} = \frac{1}{4}\Bigg\{ & \frac{4\left(\frac{2s^2}{3} + s + \frac{9}{s}\right)\left[\left(\frac{t}{3}+1\right)d\chi - \frac{1}{2}t[(1-\cos\theta)d\phi + d\tau]\right]^2}{t-s} + \\ & + \frac{4t(2t+3)\left(\left[\frac{s}{3}+1\right)d\chi - \frac{1}{2}s[(1-\cos\theta)d\phi + d\tau]\right]^2}{3(s-t)} \\ & + st\left(\sin^2\theta d\phi^2 + d\theta^2\right) + \frac{ds^2(t-s)}{\frac{2s^2}{3} + s + \frac{9}{s}} + \frac{3dt^2(s-t)}{t(2t+3)}\Bigg\} \end{aligned} \tag{5.9}$$

The reason for the subscript $\mathrm{tot}K_{\mathbb{W}P[112]}$ is that the Ricci flat metric (5.9) turns out to be defined over the total space of the canonical bundle of the (singular) projective space $\mathbb{W}P[112]$ namely on $\mathrm{tot}K_{\mathbb{W}P[112]}$.

It is a simple matter to verify that asymptotically, for $s \to -\infty$, the metric (5.9) is indeed approximatively conical, and therefore Quasi-ALE [28]. To see this, one can set $s = -\frac{2}{3}R^2$, so that

$$\mathrm{ds}^2_{\mathrm{tot}K_{\mathbb{W}P[112]}} \overset{R\to\infty}{\approx} dR^2 + R^2 \mathrm{ds}^2_{X_5} \tag{5.10}$$

at leading order in R. Since the metric is Ricci-flat Kähler, and it takes the form of a cone over a five-dimensional space, it follows that locally the five-dimensional metric $\mathrm{ds}^2_{X_5}$ is a Sasaki-Einstein metric. In appendix B we discuss the metric $\mathrm{ds}^2_{X_5}$ in more detail, showing that $X_5 = \mathbb{S}^5/\mathbb{Z}_4$, with a specific $\mathbb{Z}_4$ action.

As we will show in sect. 8, the metric induced by (5.9) on the exceptional divisor $\mathbb{W}P[112]$ is the same as the one obtained on that space while resolving a $\mathbb{C}^3/\mathbb{Z}_4$ orbifold singularity by means of the Kronheimer construction localized on the unique type III wall $\mathcal{W}_2$ displayed by its chamber structure (see sect. 6.4 of [7]).

5.2 Integration of the complex structure and the complex coordinates

In their algebraic geometry description, the varieties of the type here considered are complex threefolds $\mathcal{K}_3$ that are canonical bundles of some compact Kähler two-fold $\mathcal{D}_2$ which, on its turn, is the total space of a line-bundle over $\mathbb{P}_1$:

$$\mathcal{M}_6 = \mathcal{K}_3 \xrightarrow{\pi} \mathcal{D}_2 \xrightarrow{\widetilde{\pi}} \mathbb{P}_1 \tag{5.11}$$

This hierarchical structure implies a hierarchy in the complex coordinates that can be organized and named in the following way according with the nomenclature of [7]:

$$\begin{aligned} u &= \text{coordinate on the } \mathbb{P}_1 \text{ base of } \mathcal{D}_2 \quad ; \quad v = \text{coordinate on the fibers of } \mathcal{D}_2 \\ w &= \text{coordinate on the fibers of } \mathcal{K}_3 \end{aligned} \tag{5.12}$$

This structure is reflected in the integration of the complex structure that can be deduced from the combination of the Kähler 2-form with the metric.

5.2.1 The path to the integration

Indeed, having the metric and the Kähler form we can construct the complex structure tensor. Then we try to integrate the complex structure we have found. This is very important in order to organize the fibred structure of the manifold. First from eq. (5.2) one reads off the vielbein E^i_μ defined as:

$$\boldsymbol{E}^i \quad = \quad E^i_\mu dx^\mu \quad ; \quad x^\mu \, = \, \{s, t, \theta, \phi, \tau, \chi\} \tag{5.13}$$

The 6×6 matrix E^i_μ depends only on the s, t variables and on the angle θ (as we will see θ can be traded for the coordinate $\rho = \tan \frac{\theta}{2}$ and in the symplectic formalism it is a moment variable). The true angular variables are the phases of the three complex coordinates namely ϕ, τ, χ. As a next step one introduces the inverse vielbein which is just the matrix inverse of E^i_μ according with the definition

$$E^i_\mu E^\mu_j = \delta^i_j \tag{5.14}$$

This enables us to write the differentials of the coordinates as linear combinations of the vielbein $dx^\mu = E^\mu_j \boldsymbol{E}^i$.

THE COMPLEX STRUCTURE TENSOR IN COORDINATE INDICES $\mathbb{JW}$. Using the vielbein matrix and its inverse we can convert the frame indices of the complex structure tensor to coordinate ones and we get:

$$\mathbb{JW}_\mu{}^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{3}\Upsilon(s)^2 & -\frac{s\Upsilon(s)^2}{6+2s} \\ 0 & 0 & 0 & 0 & -\frac{1}{3}(3+t)P(t)^2 & -\frac{1}{2}tP(t)^2 \\ 0 & 0 & 0 & \csc\theta & -\tan\frac{\theta}{2} & 0 \\ -\frac{2(3+s)t\sin^2\frac{\theta}{2}}{(s-t)\Upsilon(s)^2} & -\frac{2s\sin^2\frac{\theta}{2}}{(t-s)P(t)^2} & -\sin\theta & 0 & 0 & 0 \\ -\frac{(3+s)t}{(s-t)\Upsilon(s)^2} & -\frac{s}{(t-s)P(t)^2} & 0 & 0 & 0 & 0 \\ \frac{2(3+s)(3+t)}{3(s-t)\Upsilon(s)^2} & \frac{2(3+s)}{3(t-s)P(t)^2} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

5.2.2 Integration of the autodifferentials

The matrix $\mathbb{JW}$ has three eigenvectors corresponding to the eigenvalue i and three corresponding to the eigenvalue $-$i (their complex conjugates). The three eigenvectors corresponding to i are the rows of the following matrix

$$Y^i_\mu = \begin{pmatrix} \frac{\mathrm{i}s\Upsilon(s)^2}{6+2s} & \frac{1}{2}\mathrm{i}tP(t)^2 & 0 & 0 & 0 & 1 \\ \frac{1}{3}\mathrm{i}\Upsilon(s)^2 & \frac{1}{3}\mathrm{i}(3+t)P(t)^2 & \mathrm{i}\tan\frac{\theta}{2} & 0 & 1 & 0 \\ 0 & 0 & -\mathrm{i}\csc\theta & 1 & 0 & 0 \end{pmatrix} \tag{5.16}$$

combining with the differentials $dY^i = Y^i_\mu dx^\mu$ we obtain three closed one-forms $d\left(dY^i\right) = 0$ that can be integrated to yield the three-complex variables u, v and w.

THE COORDINATE u is obtained from the integration of dY^3:

$$u = \exp \mathrm{i} \int (-\mathrm{i}\csc\theta d\theta + d\phi) = \tan\frac{\theta}{2} e^{\mathrm{i}\phi} \tag{5.17}$$

THE COORDINATE v is obtained from the integration of dY^2:

$$v = \exp \mathrm{i} \int \left(d\tau + \frac{1}{3}\mathrm{i}(3+t)P(t)^2\, dt + \mathrm{i}\tan\frac{\theta}{2}\, d\theta + \frac{1}{3}\mathrm{i}\Upsilon(s)^2\, ds\right) = \cos^2\frac{\theta}{2} H(t)\, \Psi(s) e^{\mathrm{i}\tau} \tag{5.18}$$

where we have introduced the following new functions of t and s:

$$H(t) = \exp\left(-\frac{1}{3}\int_{\mathrm{const}}^t (3+x)P(x)^2 dx\right) \quad ; \quad \Psi(s) = \exp\left(-\frac{1}{3}\int_{-\infty}^s \Upsilon(x)^2\, dx\right) \tag{5.19}$$

THE COORDINATE w is obtained from the integration of dY^1. Here we are not assisted by SU(2) invariance to define the exact coefficient in front of the differential. We choose a coefficient that appears reasonable from the result and what we obtain is either the coordinate w of other approaches or a power w^a. In the sequel comparing with the construction from the iterative procedure we will see what is the correct identification of the power a. At the beginning our educated guess suggests the use of a coefficient 4/3. So we set

$$w = \exp \mathrm{i}\frac{4}{3}\int \left(d\chi + \frac{1}{2}\mathrm{i}\, t\, P(t)^2 dt + \frac{\mathrm{i}\, s\, \Upsilon(s)^2}{6+2s}\, ds\right) = \Phi(s)K(t)e^{\mathrm{i}\frac{4}{3}\chi} \tag{5.20}$$

where we have introduced the new functions:

$$\Phi(s) = \exp\left(-\frac{2}{3}\int_{-\infty}^{s}\frac{x}{x+3}\Upsilon(x)^2 dx\right) \quad ; \quad K(t) = \exp\left(-\frac{2}{3}\int_{\mathrm{const}}^{t} x\, P(x)^2\, dx\right) \tag{5.21}$$

One necessary property that must be possessed by the function $\Phi(s)$ is:

$$\Phi(-3) = 0 \tag{5.22}$$

which defines the exceptional divisor at $w = 0$

Notice that with the ranges of the coordinates that we specified in (5.1), we see that u is a complex coordinate on a $\mathbb{P}^1$, while ν and w are complex coordinates on two copies of $\mathbb{C}$.

6 AMSY symplectic formalism and transcription of the metric in this formalism

According to the formalism introduced by Abreu [60] and developed by Martelli, Sparks and Yau [61], in the case of toric Kähler varieties of complex dimension n, one can find moment maps μ^i and angular variables Θ_i such that the Kähler 2-form takes the universal form:

$$\mathbb{K} = \sum_{i=1}^{n} d\mu^i \wedge d\Theta_i \tag{6.1}$$

At the same time there exist a function $G(\mu^i)$ of the n real moment variables, named the *symplectic potential*, such that the metric takes the following universal form:

$$\mathrm{ds}^2_{\mathrm{symp}} = G_{ij} d\mu^i d\mu^j + \left(G^{-1}\right)^{ij} d\Theta_i d\Theta_j \tag{6.2}$$

where by definition: $G_{ij} \equiv \partial_{i,j} G$ is the Hessian of the symplectic potential and $\left(G^{-1}\right)^{ij}$ is the inverse of the Hessian matrix.

In our case the three angular variables are $\boldsymbol{\Theta} = \{\phi, \tau, \chi\}$ and the Kähler form is given by $\mathbb{K}$ as defined in eq. (5.5). Transforming the pseudo angle θ to the variable ρ by setting $\theta = 2\arctan\rho$ and implementing such change of variables in the Kähler form we obtain:

$$\begin{aligned}\mathbb{K} \;=\; & \frac{1}{12}\left[3tds\wedge d\tau + \frac{6t\rho^2 ds\wedge d\phi}{1+\rho^2} - 2(3+t)ds\wedge d\chi + 3s\left(dt\wedge d\tau + \frac{2\rho^2 dt\wedge d\phi}{1+\rho^2}\right)\right.\\ & \left. -2(3+s)dt\wedge d\chi + \frac{12st\rho d\rho\wedge d\phi}{(1+\rho^2)^2}\right]\end{aligned} \tag{6.3}$$

which is compatible with eq. (6.1) if the coefficient of each of the three angular variables τ, χ , ϕ is a closed differential that can be integrated to a single new moment coordinate function of the real coordinates ρ, s, t. Hence we introduce the vector of moments:

$$\mu = \{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\} \tag{6.4}$$

and the Kähler 2-form (6.3) can be rewritten as:

$$\mathbb{K} = d\mathfrak{u} \wedge d\phi + d\mathfrak{v} \wedge d\tau + d\mathfrak{w} \wedge d\chi \tag{6.5}$$

provided we have defined the coordinate transformation:

$$\mathfrak{u} = \frac{s\,t\,\rho^2}{2+2\rho^2} \quad ; \quad \mathfrak{v} = \frac{s\,t}{4} \quad ; \quad \mathfrak{w} = \frac{1}{6}\left(-3t - s(3+t)\right) \tag{6.6}$$

The unique inverse transformation of the above coordinate change is the following one:

$$\begin{array}{rclcrcl} \rho & = & \frac{\sqrt{\mathfrak{u}}}{\sqrt{-\mathfrak{u}+2\mathfrak{v}}} & ; & t & = & \frac{1}{6}\left(-4\mathfrak{v} - 6\mathfrak{w} + \sqrt{-144\mathfrak{v} + (4\mathfrak{v}+6\mathfrak{w})^2}\right) \\ & & & & s & = & \frac{1}{3}\left(-2\mathfrak{v} - 3\mathfrak{w} - \sqrt{4(-9+\mathfrak{v})\mathfrak{v} + 12\mathfrak{v}\mathfrak{w} + 9\mathfrak{w}^2}\right) \end{array} \tag{6.7}$$

The new real coordinates are named $\mathfrak{u}$,$\mathfrak{v}$,$\mathfrak{w}$ with gothic letters since they are the symplectic counterparts of the complex coordinates u, v, w yet, differently from the latter, we do not need the complex structure to find them and hence they are independent from the metric.

6.1 Transcription of the metric in the toric symplectic form

At this point we try to rewrite the metric depending on the two functions:

$$M(s,t) \equiv \sqrt{\frac{s-t}{3+s}}\,\Upsilon(s) \quad ; \quad \Phi(s,t) \equiv \sqrt{t-s}\,P(t) \tag{6.8}$$

in the symplectic form (6.2). Setting:

$$M(s,t) \;=\; \mathfrak{M}(\mathfrak{v},\mathfrak{w}) \equiv \mathfrak{M} \quad ; \quad \Phi(s,t) \;=\; \mathfrak{F}(\mathfrak{v},\mathfrak{w}) \equiv \mathfrak{F} \quad ; \quad \Omega \;\equiv\; 4(-9+\mathfrak{v})\mathfrak{v} + 12\mathfrak{v}\mathfrak{w} + 9\mathfrak{w}^2 \tag{6.9}$$

we easily derive that ds^2_{ort} takes the form (6.2) with the following matrix $\mathcal{G}_{ij}$:

$$\begin{array}{rcl} \mathcal{G}_{11} & = & -\dfrac{\mathfrak{v}}{\mathfrak{u}^2 - 2\mathfrak{u}\mathfrak{v}} \\ \mathcal{G}_{12} & = & \dfrac{1}{\mathfrak{u} - 2\mathfrak{v}} \\ \mathcal{G}_{13} & = & 0 \\ \mathcal{G}_{22} & = & \dfrac{1}{9}\left[-\dfrac{9\mathfrak{u}}{\mathfrak{u}\mathfrak{v} - 2\mathfrak{v}^2} + \dfrac{\mathfrak{F}^2\left(-2\mathfrak{v} - 3\mathfrak{w} + \sqrt{\Omega} + 9\right)^2}{\Omega} + \dfrac{\mathfrak{M}^2\left(2\mathfrak{v} + 3\mathfrak{w} + \sqrt{\Omega} - 9\right)^2}{\Omega}\right] \\ \mathcal{G}_{23} & = & \dfrac{1}{6\Omega}\left\{\mathfrak{F}^2\left[8\mathfrak{v}^2 + \mathfrak{v}\left(24\mathfrak{w} - 4\sqrt{\Omega} - 54\right) - 3(2\mathfrak{w} - 3)\left(\sqrt{\Omega} - 3\mathfrak{w}\right)\right]\right. \\ & & \left. +\mathfrak{M}^2\left[8\mathfrak{v}^2 + \mathfrak{v}\left(24\mathfrak{w} + 4\sqrt{\Omega} - 54\right) + 3(2\mathfrak{w} - 3)\left(3\mathfrak{w} + \sqrt{\Omega}\right)\right]\right\} \\ \mathcal{G}_{33} & = & \dfrac{1}{16\Omega}\left[4\mathfrak{F}^2\left(-2\mathfrak{v} - 3\mathfrak{w} + \sqrt{\Omega}\right)^2 + 4\mathfrak{M}^2\left(2\mathfrak{v} + 3\mathfrak{w} + \sqrt{\Omega}\right)^2\right] \end{array} \tag{6.10}$$

It remains to be seen if we are able to retrieve the symplectic potential from which the above matrix is obtained through double derivatives.

With some integrations and some educated guesses we find that the form (6.10) of the matrix can be reproduced if we write the symplectic potential as follows:

$$G(\mathfrak{u},\mathfrak{v},\mathfrak{w}) \;=\; G_0(\mathfrak{u},\mathfrak{v}) + \mathcal{G}(\mathfrak{v},\mathfrak{w}) \tag{6.11}$$

where

$$G_0(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}\left(-\mathfrak{u}+\mathfrak{u}\log\mathfrak{u}\right) - \mathfrak{v}\log\mathfrak{v} + \left(-\frac{\mathfrak{u}}{2}+\mathfrak{v}\right)\log(-\mathfrak{u}+2\mathfrak{v}) \tag{6.12}$$

and where $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ is some function of the two fibre coordinates $\mathfrak{v},\mathfrak{w}$ only. With this choice the matrix $G_{\rm ij}$ becomes:

$$G_{ij} = \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2-2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u}-2\mathfrak{v}} & 0 \\ \frac{1}{\mathfrak{u}-2\mathfrak{v}} & -\frac{\mathfrak{u}}{\mathfrak{u}\mathfrak{v}-2\mathfrak{v}^2}+\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w}) & \mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w}) \\ 0 & \mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w}) & \mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w}) \end{pmatrix} \tag{6.13}$$

and the full-fledged expression of the line element can be obtained by substitution. Comparing the obtained result with eq. (6.10) we easily see that the functions $\mathfrak{M}(\mathfrak{v},\mathfrak{w}) = M(s,t)$ and $\mathfrak{F}(\mathfrak{v},\mathfrak{w}) = \Phi(s,t)$ can be expressed in terms of the derivatives $\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w})$, $\mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w})$, but in order to avoid other functions we get a second order differential constraint on the symplectic potential $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ that relates its mixed derivatives to $\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w})$, $\mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w})$. This differential is expressed in a simpler way by means of the original coordinates s,t. We shall presently derive it. We anticipate that its solution very strongly limits the possibilities so that it has to be discarded. In other words we have to accept a generic function $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ and try to match it with the boundary conditions on the exceptional divisor.

6.2 Orthotoric separation of variables and the symplectic potential

In order to compare the generic metric in symplectic formalism provided by the symplectic potential displayed in eq.s (6.11), (6.12) with the following two-function metric[22]:

$$\begin{aligned} \mathrm{ds}^2_{2fun} &= \frac{1}{4}st\left(d\phi^2\sin^2\theta+d\theta^2\right) + \frac{1}{M(s,t)^2}\left[d\chi - \frac{1}{2}t\left(d\phi(1-\cos(\theta))+d\tau-\frac{2d\chi}{3}\right)\right]^2 \\ &+ \frac{1}{\Phi(s,t)^2}\left[d\chi-\frac{1}{2}s\left(d\phi(1-\cos\theta)+d\tau-\frac{2d\chi}{3}\right)\right]^2 + \frac{1}{4}dt^2\Phi(s,t)^2+\frac{1}{4}ds^2M(s,t)^2 \end{aligned} \tag{6.14}$$

we make the following steps. First we regard the function $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ as a function only of t and s, as it is evident from the transformation rule (6.7), and we write: $\mathcal{G}(\mathfrak{v},\mathfrak{w}) \equiv \Gamma(t,s)$. By means of the transformation (6.7) we can rewrite the generic metric (6.2) produced by the symplectic potential (6.11-6.12) in terms of the variables s,t, instead of $\mathfrak{v},\mathfrak{w}$. The result coincides with ds^2_{2fun} as given in eq. (6.14) if the following conditions hold true:

$$\begin{aligned} &\frac{\partial^2}{\partial s^2}\Gamma(t,s) \;=\; \frac{1}{4}\,M(s,t)^2 \quad ; \quad \frac{\partial^2}{\partial t^2}\Gamma(t,s) \;=\; \frac{1}{4}\,\Phi(s,t)^2 \\ &\frac{\partial}{\partial s}\Gamma(t,s) \;-\; \frac{\partial}{\partial t}\Gamma(t,s) + (s-t)\frac{\partial^2}{\partial s\,\partial t}\Gamma(t,s) \;=\; 0 \end{aligned} \tag{6.15}$$

The first two equations in (6.15) just provide the identification of the two functions $M(s,t)$ and $\Phi(s,t)$ in terms of second order derivatives of the symplectic potential. On the other hand the last equation of (6.15) is a very strong constraint on the function $\Gamma(t,s)$ which severely restricts the available choices of $\Gamma(t,s)$.

[22] At this level we do not require $M(s,t)$ and $\Phi(s,t)$ to have the specific form of eq. (6.8)

6.3 The symplectic potential of the Ricci-flat orthotoric metric on $\mathrm{tot}K_{\mathbb{W}P[112]}$

In the case of the canonical bundle $\mathrm{tot}K_{\mathbb{W}P[112]}$, whose Ricci–flat metric is given by eq. (5.9), eq.s (5.8) imply

$$
\begin{aligned}
\Gamma^{(0,2)}(t,s) &= \frac{(s-t)\Upsilon(s)^2}{4(s+3)} = -\frac{s(s-t)}{4(s+3)\left(\frac{2s^2}{3}-s+3\right)} \quad \Rightarrow \quad \Upsilon(s)^2 = \frac{s}{\left(\frac{2s^2}{3}-s+3\right)} \\
\Gamma^{(2,0)}(t,s) &= \frac{1}{4}P(t)^2(t-s) = \frac{s-t}{4\left(\frac{2t^2}{3}+t\right)} \quad \Rightarrow \quad P(t)^2 = -\frac{1}{\frac{2}{3}t\,(t+3)}
\end{aligned} \tag{6.16}
$$

By means of two double integrations and modulo linear functions in s,t (they are irrelevant for the metric) we determine the explicit form of the potential $\Gamma(t,s)$:

$$
\begin{aligned}
\Gamma_{\mathbb{W}P[112]}(t,s) &= \frac{1}{224}\Big\{-7\left[(3t-st+3s)\log(2s^2-3s+9)+2(s+3)(t+3)\log(s+3)-8st\log t\right. \\
&\left.+2(2s+3)(2t+3)\log(2t+3)\right]-6\sqrt{7}(st+s+t-6)\arctan\frac{3-4s}{3\sqrt{7}}\Big\}
\end{aligned} \tag{6.17}
$$

The function $\Gamma_{\mathbb{W}P[112]}(t,s)$ satisfies by construction the differential constraint encoded in the third of eq.s (6.16). Using the transformation rule (6.7) we can rewrite it as a function of the symplectic variables $\mathfrak{v},\mathfrak{w}$. In this way we arrive at the following symplectic potential where we have used the liberty of adding linear functions of $\mathfrak{v}$ or $\mathfrak{w}$ to obtain the most convenient form of its reduction to the exceptional divisor, located at $\mathfrak{w}=\frac{3}{2}$. The function

$$
\begin{aligned}
\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w}) &= \frac{1}{224}\Big\{7\Big[6(2\mathfrak{w}-3)\log\left(-\sqrt{(2\mathfrak{v}+3\mathfrak{w})^2-36\mathfrak{v}}-2\mathfrak{v}-3\mathfrak{w}+9\right) \\
&+16\mathfrak{v}\log\left(\sqrt{(2\mathfrak{v}+3\mathfrak{w})^2-36\mathfrak{v}}-2\mathfrak{v}-3\mathfrak{w}\right)^2 \\
&-2(8\mathfrak{v}-12\mathfrak{w}+9)\log\left(\sqrt{(2\mathfrak{v}+3\mathfrak{w})^2-36\mathfrak{v}}-2\mathfrak{v}-3\mathfrak{w}+\frac{9}{2}\right) \\
&+2(4\mathfrak{v}+3\mathfrak{w})\log\left(\frac{1}{567}\left[4\sqrt{(2\mathfrak{v}+3\mathfrak{w})^2-36\mathfrak{v}}+8\mathfrak{v}+12\mathfrak{w}+9\right]^2+1\right)\Big] \\
&-4\sqrt{7}(4\mathfrak{v}-3(\mathfrak{w}+3))\arctan\left(\frac{4\sqrt{(2\mathfrak{v}+3\mathfrak{w})^2-36\mathfrak{v}}+8\mathfrak{v}+12\mathfrak{w}+9}{9\sqrt{7}}\right) \\
&-(8\mathfrak{v}+9)\log\frac{34359738368}{823543}+2\sqrt{7}(8\mathfrak{v}-27)\arctan\frac{5}{\sqrt{7}}\Big\}
\end{aligned} \tag{6.18}
$$

is expressed in terms of elementary transcendental functions, yet it has the remarkable property of satisfying the Monge-Ampère equation for Ricci-flatness, so that it may be called “the miraculous function”. On the exceptional divisor it reduces to

$$
\mathcal{D}_{\mathbb{W}P[112]}(\mathfrak{v}) \equiv \mathcal{G}_{\mathbb{W}P[112]}\left(\mathfrak{v},\tfrac{3}{2}\right) = \frac{1}{16}\left[8\mathfrak{v}\log\left(16\mathfrak{v}^2\right)+(9-8\mathfrak{v})\log\left(\frac{9}{2}-4\mathfrak{v}\right)\right]. \tag{6.19}
$$

7 Kähler metrics on Hirzebruch surfaces and their canonical bundles

For the case of the canonical bundle on $\mathbb{F}_2$, which is the complete resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity, we have additional information that is relevant and inspiring for the general case.

Let us summarize the main points. According to the results of [7] there is a well adapted system of complex coordinates that arise from the toric analysis of $\mathbb{C}^3/\mathbb{Z}_4$ and of its resolution. These coordinates are named as follows: $z_i = \{u, v, w\}$ and are defined on a dense open chart reaching all components of the exceptional divisor. Their interpretation was already anticipated in eq. (5.12) and it is the following. The coordinate w spans the fibers in the canonical bundle $Y \stackrel{\pi}{\longrightarrow} \mathbb{F}_2$ while u, v span a dense open chart for the base manifold (*i.e.* the compact component $\mathbb{F}_2$ of the exceptional divisor $\mathcal{ED}$). In particular since $\mathbb{F}_2$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, namely $\mathbb{F}_2 \stackrel{\pi}{\longrightarrow} \mathbb{P}^1$, the coordinate u is a standard Fubini-Study coordinate for the base $\mathbb{P}^1$ while v spans a dense open chart of the fibre $\mathbb{P}^1$. This set of coordinates can be used for any $\mathbb{F}_n$ Hirzebruch surface with $n \geq 1$. The action of the isometry group (1.8) on these coordinates was described in [7] and it is as follows:

$$
\begin{array}{rclcl}
\forall \mathbf{g} & = & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2) & : & \mathbf{g}\,(u,v,w) \,=\, \left(\dfrac{a\,u+b}{c\,u+d}, \quad v\,(c\,u+d)^n\,, \quad w\right) \\
\forall \mathbf{g} & = & \exp[\mathrm{i}\,\theta_1] \in \mathrm{U}(1)_v & : & \mathbf{g}\,(u,v,w) \,=\, (u, \quad \exp[\mathrm{i}\,\theta_1]\,v \quad w) \\
\forall \mathbf{g} & = & \exp[\mathrm{i}\,\theta_2] \in \mathrm{U}(1)_w & : & \mathbf{g}\,(u,v,w) \,=\, (u, \quad v, \quad \exp[\mathrm{i}\,\theta_2]\,w)
\end{array}
\tag{7.1}
$$

The above explicit action of the isometry group on the u, v, w coordinates suggests the use of an invariant real combination

$$
\varpi_n \;\equiv\; \left(1+ \mid u \mid^2\right)^n \; \mid v \mid^2 \tag{7.2}
$$

and the assumption that the Kähler potential $\mathcal{K}_{\mathbb{F}_n}$ of the Kähler metric $\mathbf{g}_{\mathbb{F}_n}$ should be a function (up to trivial terms $\mathrm{Re}f(z)$) only of ϖ_n:

$$
\mathcal{K}_{\mathbb{F}_n} \;=\; G_n(\varpi_n) \tag{7.3}
$$

The function $G_n(\varpi_n)$ should also depend on two parameters (we name them ℓ,α) which are associated to the volumes of the two homology cycles of $\mathbb{F}_n$, respectively named C_1 and C_2 that also form a basis for the homology group of the total space Y, namely the canonical bundle on $\mathbb{F}_n$. Indeed the homology of Y coincides with the homology of the base manifold $\mathbb{F}_n$.

Introducing the Kähler two form:

$$
\mathbf{K}_{\mathbb{F}_n} \;\equiv\; \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\mathcal{K}_{\mathbb{F}_n} \tag{7.4}
$$

we need to find:

$$
\int_{C_1} \mathbf{K}_{\mathbb{F}_n} \;=\; \frac{9}{16}\,\alpha\,\ell \quad ; \quad \int_{C_2} \mathbf{K}_{\mathbb{F}_n} \;=\; \frac{9}{16}\,(2+\alpha)\,\ell \tag{7.5}
$$

where ℓ is a dimensionful parameter providing the scale and α is some dimensionless parameter parameterizing the ratio between the two volumes. The two toric cycles $C_{1,2}$ are respectively defined by the following two equations:

$$
C_1 \;\Leftrightarrow\; v=0 \quad ; \quad C_2 \;\Leftrightarrow\; u=0 \tag{7.6}
$$

As pointed out in [7], in addition to the above two properties of the Kähler form, if we consider the Ricci two-form of the Kähler metric on $\mathbb{F}_n$

$$
\mathbf{Ric}_{\mathbb{F}_n} \;=\; \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log\left[\det\left(\mathbf{g}^{\mathbb{F}_n}\right)\right] \quad ; \quad \mathbf{g}^{\mathbb{F}_n}_{ij^\star} \;=\; \partial_i\partial_{j^\star}\,\mathbf{K}_{\mathbb{F}_n} \qquad i=1,2 \quad j^\star = 1^\star, 2^\star \tag{7.7}
$$

we must find:

$$
\int_{C_1} \mathbf{Ric}_{\mathbb{F}_n} \;=\; 2-n \quad ; \quad \int_{C_2} \mathbf{Ric}_{\mathbb{F}_n} \;=\; 2 \tag{7.8}
$$

It appears that eq.s (7.5-7.8) are strong constraints on the function $G_n(\varpi_n)$. It is interesting to see how they are realized in the metric on $\mathbb{F}_2$ obtained from the Kronheimer construction. We will show this below.

7.1 The metric on $\mathbb{F}_2$ induced by the Kronheimer construction

In [7], relying on the Kronheimer construction, we have constructed an analytically defined Kähler metric on the total space of the canonical bundle of $\mathbb{F}_2$. The Kähler potential has only an implicit definition as the largest real root of a sextic equation. Yet its reduction to the compact exceptional divisor, which is indeed the 2nd Hirzebruch surface, is explicit and the Kähler potential of this metric can be exhibited in closed analytic form. We think that this information is very important for the comparison between the parameters of the Ricci-flat metric appearing in supergravity with those emerging in the Kronheimer construction that are the Fayet Iliopoulos parameters of the dual gauge theory.

Following the chamber structure discussed in [7] we choose the chamber VI defined by the following inequalities on the three Fayet Iliopoulos parameters $\zeta_{1,2,3}$:

$$\zeta_1 - \zeta_2 - \zeta_3 < 0 \quad ; \quad -\zeta_1 + \zeta_2 - \zeta_3 > 0 \quad ; \quad -\zeta_1 - \zeta_2 + \zeta_3 < 0 \tag{7.9}$$

and chamber VIII, defined instead by the following ones:

$$\zeta_1 - \zeta_2 - \zeta_3 < 0 \quad ; \quad -\zeta_1 + \zeta_2 - \zeta_3 < 0 \quad ; \quad -\zeta_1 - \zeta_2 + \zeta_3 < 0 \tag{7.10}$$

Inside those two chambers we make the choice:

$$\zeta_1 \,=\, \zeta_3 \,=\, r \quad ; \quad \zeta_2 \,=\, (2+\alpha)r \quad ; \qquad r > 0 \tag{7.11}$$

For $\alpha > 0$ we are in chamber VI, while for $\alpha < 0$ we are in chamber VIII. For $\alpha = 0$ we are instead on the wall where the non singular variety:

$$Y \,\equiv\, \mathrm{tot}K_{\mathbb{F}_2} \tag{7.12}$$

degenerates in

$$Y_3 \,\equiv\, \mathrm{tot}K_{\mathbb{W}P[112]} \tag{7.13}$$

denoting by $\mathrm{tot}K_{\mathcal{M}}$ the total space of the canonical bundle of a Kähler manifold (or orbifold) $\mathcal{M}$.

The solution of the moment map equations for the two independent moment maps reduced to the exceptional divisor by performing the limit $w \to 0$ is the following one:

$$T_1 = T_3 \,=\, \sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{2(\alpha+2)\sqrt{\varpi/v\overline{v}}}}\,;\, T_2 \,=\, \frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4}{2\alpha^2+6\alpha+4} \tag{7.14}$$

The complete Kähler potential of the quotient is made of two addends, the pull-back on the constrained surface of the Kähler potential of the flat ambient metric plus the logarithmic term:

$$\mathcal{K}_{quotient} \,=\, \mathcal{K}_0 \,+\, \underbrace{\zeta_I \mathfrak{C}^{IJ}\,\log T_J}_{\mathcal{K}_{log}} \quad ; \quad \mathfrak{C}^{IJ} \,=\, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{7.15}$$

In the present case we explicitly find:

$$\mathcal{K}_0 \,=\, 2\frac{\alpha\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+2\varpi+1\right)+\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha^2+3\varpi}{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi} \tag{7.16}$$

and

$$\mathcal{K}_{log} \;=\; 2(\alpha+1)\log\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4}{2\alpha^2+6\alpha+4} - 2\alpha\log\sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{2(\alpha+2)\sqrt{\varpi/v\overline{v}}}} \tag{7.17}$$

By explicit calculation we were able to verify that the Kähler potential of the quotient $\mathcal{K}_{quotient}$ yields a metric satisfying all the constraints (7.5-7.8). We show this in section 8.3.

8 Reduction to the exceptional divisor

In this section we consider the reduction to the exceptional divisor for a generic metric of the class described in section 5, emphasizing that the Kähler metric induced on the divisor is completely determined by the real function $P(t)$ of the real variable t. We carefully consider what are the differential constraints on such a function required by the topology and complex structure of the second Hirzebruch surface $\mathbb{F}_2$ showing that they are all met by the $P(t)$ function that one obtains by localizing the generalized Kronheimer construction of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity resolution on the exceptional divisor.

8.1 The reduction

The reduction to the exceptional divisor is obtained in the Kähler form and in the metric by setting $s = -3$. The Kähler form on the divisor is the following one

$$\mathbb{K}_{\mathcal{ED}} = \frac{1}{12}\left(-9\,t\,\sin\theta d\theta\wedge d\phi - 9\,dt\wedge d\tau + 9\,(\cos\theta - 1)\,dt\wedge d\phi\right) \tag{8.1}$$

while the metric is the following one:

$$\mathrm{ds}^2_{\mathcal{ED}} = -\frac{3\,t}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{4}(t+3)P(t)^2dt^2 + \frac{9[d\tau + (1-\cos\theta)d\phi]^2}{4(t+3)P(t)^2} \tag{8.2}$$

and it is completely determined by the function $P(t)$. For the choice:

$$P(t) = \left(-\frac{2}{3}t^2 - t\right)^{-\frac{1}{2}} \tag{8.3}$$

it is the metric on the orbifold $\mathbb{W}P[112]$ while for other choices of $P(t)$, obtainable from the Kronheimer construction, $\mathrm{ds}^2_{\mathcal{ED}}$ can indeed be a good Kähler metric on the second Hirzebruch surface $\mathbb{F}_2$.

From eq. (8.1) specifying the Kähler 2-form of the exceptional divisor and eq. (8.2) providing its Kähler metric, we immediately work out also the complex structure tensor that has the following appearence:

$$\mathbb{J}_{\mathcal{ED}} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{3}(t+3)P(t)^2 \\ 0 & 0 & \csc\theta & -\tan\frac{\theta}{2} \\ \frac{6\sin^2\frac{\theta}{2}}{(t+3)P(t)^2} & -\sin\theta & 0 & 0 \\ \frac{3}{(t+3)P(t)^2} & 0 & 0 & 0 \end{pmatrix} \tag{8.4}$$

8.2 Topology and the functions of the t coordinate

We have two important informations on the topology of $\mathbb{F}_2$, which provide an extremely selective test in order to know whether a certain metric is indeed defined on $\mathbb{F}_2$ or on some different twofold, may be degenerate. The tests are related with the integrals of the Kähler 2-form $\mathbb{K}$ and of the Ricci 2-form $\mathbb{R}\mathrm{ic}$ on the two toric curves $C_{1,2}$ respectively defined by the vanishing of either coordinate (u, v)

$$C_1 = \{v = 0\} \quad ; \quad C_2 = \{u = 0\} \tag{8.5}$$

Indeed, as we illustrated in section 7 we must find

$$\int_{C_1}\mathbb{K} \neq \infty \quad ; \quad \int_{C_2}\mathbb{K} \neq \infty \quad ; \quad \int_{C_1}\mathbb{K} \neq 0 \quad ; \quad \int_{C_2}\mathbb{K} \neq 0 \quad ; \quad \int_{C_1}\mathbb{R}\mathrm{ic} = 0 \quad ; \quad \int_{C_2}\mathbb{R}\mathrm{ic} = 2 \tag{8.6}$$

The explicit reduction of the Kähler form $\mathbb{K}_{\mathbb{F}_2}$ to the two cycles C_1 and C_2 is very simple when $\mathbb{K}_{\mathbb{F}_2}$ is written in the basis of the real coordinates (t,θ,τ,ϕ). Indeed in order to set $v = 0$ we have just to look for the zeros of the above defined function $H(t)$ that depends by integration from $P(t)$. Let us suppose that $H(-|t_{max}|) = 0$. We obtain the reduction of the Kähler form to the cycle C_1 by setting $t = -|t_{max}|$ = const < 0, while we get the reduction to the cycle C_2 by setting $\theta = 0$.

$$\mathbb{K}\,|_{C_1} = \frac{3}{4}\,|t_{max}|\,\sin\theta\, d\theta \wedge d\phi \quad ; \quad \mathbb{K}\,|_{C_2} = -\frac{3}{4} dt \wedge d\tau \tag{8.7}$$

Hence we see that in order to get $\mathbb{F}_2$ as exceptional divisor we need two conditions, that are necessary, although not sufficient.

1. $|t_{max}| \neq 0$
2. the range of the coordinate t must be finite $[-|t_{max}|, -|t_{min}|]$ in order to get a finite size for the cycle C_2

If the zero of the function $H(t)$ is at t=0 we immediately know that there is a degeneration and this is indeed the case of $\mathbb{W}P[112]$.

If we integrate the complex structure of the exceptional divisor displayed in eq. (8.4) with the same method we used for the whole 6-dimensional space, we find that the coordinate u is exactly the same as in eq. (5.17), while for v we find:

$$v \;=\; H(t)\cos^2\frac{\theta}{2} e^{\mathrm{i}\tau} \tag{8.8}$$

Comparison with the result for v in the entire space (eq.s (5.18-5.19)) tells us that the function $\Psi(s)$ must be finite and non vanishing at $s = -3$ in order to have a consistent reduction to the divisor:

$$\Psi(-3) = 1 \quad ; \quad \Phi(-3) = 0 \tag{8.9}$$

The normalization $\Psi(-3) = 1$ can always be obtained by an irrelevant rescaling in the definition of v if -3 is not a zero of $\Psi(s)$ while it must be a zero of $\Phi(s)$.

8.2.1 Interpretation of the function $H(t)$

From the explicit integration of the complex structure we obtain a very important interpretation of the function $H(t)$ in relation with the complex Kähler geometry of the exceptional divisor. Since the Kähler metric on this two-fold has isometry SU(2) $\times$ U(1), SU(2) acting on the u variable by linear fractional transformation and on v by multiplication with the u-compensator $(cu + d)^2$, as described in eq.s (7.1), the Kähler potential $\mathcal{K}$ can be a function only of the invariant combination $\varpi \equiv \varpi_2$ defined in eq. (7.2). Relying on the representation of u and v derived from the integration of the complex structure we easily obtain:

$$\varpi \;=\; \cos^4\frac{\theta}{2}\left(\tan^2\frac{\theta}{2} + 1\right)^2 H(t)^2 \;=\; H(t)^2 \tag{8.10}$$

It follows that:

$$t \;=\; H^{-1}(\sqrt{\varpi}) \tag{8.11}$$

where H^{-1} denotes the inverse function. Since the range of $\sqrt{\varpi}$ is $[0, \infty]$, it is necessary that the inverse function H^{-1} maps the semi-infinite interval $[0, \infty]$ in a finite one $[-|t_{max}|, -|t_{min}|]$ defined by:

$$-\,|t_{max}| \;=\; \lim_{\varpi\to 0} H^{-1}(t) \quad ; \quad -|t_{min}| \;=\; \lim_{\varpi\to\infty} H^{-1}(t) \tag{8.12}$$

8.2.2 Topological constraints on the function $P(t)$

Given the above topology results characterizing the second Hirzebruch surface and considering the metric of the divisor as given in eq. (8.2) and its Kähler form (8.1) we immediately obtain the conditions on the function $P(t)$. Indeed, while calculating the Ricci form we can specify integral differential conditions on $P(t)$ from the values of its periods mentioned above. We know the explicit form of the complex structure on the exceptional divisor that is obtained by reduction to $s = -3$ of the complex structure pertaining the full 6-dimensional manifold $\mathcal{M}_6$. The complex structure of the exceptional divisor was displayed in eq. (8.4). The Ricci form can be calculated by setting its antisymmetric components equal to $\mathbb{R}\mathrm{ic}_{ij} = J_i{}^k R_{\mathrm{kj}}$ where R_{kj} is the standard Ricci tensor. In this way we obtain the following general result that exclusively depends on the function $P(t)$:

$$\mathbb{R}\mathrm{ic}_{\mathcal{ED}} \;=\; \mathfrak{A}(t)\sin\theta\, d\theta\wedge d\phi + \mathfrak{B}(t)\,\sin^2\frac{\theta}{2}\, dt\wedge d\phi \;+\; \mathfrak{C}(t) dt\wedge d\tau \tag{8.13}$$

where $\mathfrak{A}(t), \mathfrak{B}(t), \mathfrak{C}(t)$ are functions of the t-variable expressed as rational functions of $P(t)$ and its first and second derivative with simple t-dependent coefficient. We do not write them explicitly for shortness. Then the Ricci 2-form can be easily localized on the two cycles C_1 and C_2, yielding:

$$\mathbb{R}\mathrm{ic}|_{C_1} \;=\; \mathfrak{A}(-|t_{max}|)\,\sin\theta\, d\theta\wedge d\phi \quad ; \quad \mathbb{R}\mathrm{ic}|_{C_2} \;=\; \mathfrak{C}(t)\, dt\wedge d\tau \tag{8.14}$$

Hence, in order to realize the second Hirzebruch surface not only the range of t must have finite extrema $[-|t_{max}|,-|t_{min}|]$ but we should also have:

$$\mathfrak{A}(-|t_{max}|) = 0 \quad ; \quad \int_{-|t_{min}|}^{-|t_{max}|} \mathfrak{C}(t)\, dt = 2 \tag{8.15}$$

8.2.3 The relation between the function $P(t)$ and the Kähler potential $\mathcal{K}(\varpi)$ of the exceptional divisor

Our goal is that of determining a Ricci-flat metric on the canonical bundle $\mathrm{tot}K_{\mathbb{F}_2}$, starting from a given *bona fide* Kähler metric on the second Hirzebruch surface, described in terms of the real variables t, θ, τ, ϕ. In the complex description, any Kähler metric is determined by a suitable Kähler potential; given the isometries and their realization on the chosen complex coordinates u, v, the Kähler potential for the $\mathbb{F}_2$ surface is a real function of the invariant combination ϖ defined in eq. (7.2) which we generically denote $\mathcal{K}(\varpi)$. Therefore it is important to determine the relation between the real variables and the standard complex ones at the same time with the relation between the Kähler potential $\mathcal{K}(\varpi)$ and the function $P(t)$ which determines the metric in the real variables. In this respect the essential point to be stressed is that the relation between the real variables and the complex ones is not universal and fixed once for all, rather it depends on the choice of the Kähler potential or viceversa of the function $P(t)$. Hence it is convenient to introduce a name for the inverse function:

$$H^{-1}\left(\sqrt{\varpi}\right) = G_T(\varpi) \tag{8.16}$$

and find its differential relation with the Kähler potential which follows from a comparison between the metric as determined in complex Kähler geometry from $\mathcal{K}(\varpi)$ and as written in real variables. For convenience we rewrite the general real form of the metric on the exceptional divisor in the following more compact way

$$\mathrm{ds}^2_{\mathcal{ED}} \;=\; -\frac{3}{4}\left(t\,\underbrace{[d\theta^2 + d\phi^2\sin^2\theta]}_{\text{metric on }\mathbb{P}^1} + F(t)\,dt^2 \;+\; \frac{1}{F(t)}\quad \underbrace{[d\tau + (1-\cos\theta)d\phi]}_{\text{connection on the U(1) bundle}}{}^2\right) \quad ; \quad F(t) \;\equiv\; -\frac{1}{3}(t{+}3)P(t)^2 \tag{8.17}$$

which clearly displays the fibred structure of the exceptional divisor.

Next we convert the metric in eq. (8.17) using the substitution rule

$$t = G_T(\varpi), \quad \theta = 2\arctan\sqrt{u\bar{u}}, \quad \tau = -\frac{1}{2}\mathrm{i}\,\log\left(\frac{v}{\bar{v}}\right), \quad \phi = -\frac{1}{2}\mathrm{i}\log\left(\frac{u}{\bar{u}}\right) \tag{8.18}$$

In this way we transform the metric (8.17) to the complex coordinates u, v and we compare it with the generic metric obtained from a generic Kähler potential $\mathcal{K}(\varpi)$. We find that the two metrics coincide provided the following two conditions are satisfied:

$$t \equiv G_T(\varpi) = -\frac{2}{3}\varpi\partial_\varpi\mathcal{K}(\varpi) \quad ; \quad P(t) = \pm\frac{3}{2\sqrt{-\varpi\left(3-\frac{2}{3}\varpi\partial_\varpi\mathcal{K}(\varpi)\right)\left(\partial_\varpi\mathcal{K}(\varpi)+\varpi\partial_{\varpi,\varpi}\mathcal{K}(\varpi)\right)}} \tag{8.19}$$

Given the Kähler potential $\mathcal{K}(\varpi)$, which is supposed to depend also on a deformation parameter, the above equation (8.19) allows to rewrite the same metric in real coordinates, provided one is able to invert the first formula, namely, to find ϖ as a function of t and of the deformation parameter α.

8.3 The Kronheimer Kähler potential for the $\mathbb{F}_2$ surface and its associated $P(t)$ function

From the Kronheimer construction of the $\mathbb{C}^3/\mathbb{Z}_4$ resolution reduced to the exceptional divisor we have the Kähler potential derived in section 7.1. The result obtained in eq.s (7.15,7.16,7.17) can be summarized writing the following general form of the Kähler potential:

$$\begin{aligned}\mathcal{K}_{\mathbb{F}_2}(\varpi,\alpha) \;=\; & -\frac{9}{16}\Bigg\{-4(\alpha+1)\log\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4\right)\\ & -\frac{4\left[\alpha\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+2\varpi+1\right)+\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha^2+3\varpi\right]}{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}\\ & +4\alpha\log\sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{\sqrt{\varpi}}}+8+16\log 2\Bigg\}\end{aligned} \tag{8.20}$$

where for additional convenience we have changed the overall normalization of the metric multiplying by a 9/8 factor, have disregarded the irrelevant addends proportional to $\log[v\,\bar{v}]$ and have added a convenient constant addend. For α=0 the surface described by the Kähler metric corresponding to the potential (8.20) degenerates into the singular $\mathbb{W}P[112]$ while for other values of α such that $0 <\mid \alpha \mid< 1$ we have a metric on a smooth $\mathbb{F}_2$ surface.

8.3.1 The degenerate case $\mathbb{W}P[112]$

It is interesting to see how we recover the degenerate case $\mathbb{W}P[112]$ from the general case. Setting α=0 we obtain:

$$\begin{aligned}\mathcal{K}_{\mathbb{W}P[112]}(\varpi) &= \frac{9}{4}\left[\frac{3\varpi+\sqrt{\varpi(\varpi+8)}}{\varpi+\sqrt{\varpi(\varpi+8)}}+\log\left(\varpi+\sqrt{\varpi(\varpi+8)}\right)-2-4\log 2\right]\\ t &= -\frac{3\varpi^2\left(\varpi+\sqrt{\varpi(\varpi+8)}+8\right)}{\sqrt{\varpi(\varpi+8)}\left(\varpi+\sqrt{\varpi(\varpi+8)}\right)^2} \quad\Rightarrow\quad \varpi=\frac{8t^2}{3(2t+3)}\end{aligned} \tag{8.21}$$

This implies that the interval [0,∞] of ϖ is mapped into the interval [0,-3/2] and this suffices to guarantee that the cycle C_1 is contracted to zero as we have already explained. Finally for the function $P(t)$, using the above general formulae we get:

$$P(t) = \sqrt{\frac{-3}{2t^2+3t}} \tag{8.22}$$

8.3.2 The smooth $\mathbb{F}_2$ case

First we can verify that when α is either -1 or -2, the surface degenerates, as the metric depends only on the variable u and no longer on v. Using the formula (8.19) we can calculate t and $P(t)$. We find the following relatively complicated answer:

$$\begin{aligned}
t &= G_T(\varpi) = \frac{N_T}{D_T} \\
N_T &= -3\Big\{\alpha^4 + \alpha\varpi\Big[3\varpi\Big(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+16\Big)+8\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+3\varpi^2\Big] \\
&\quad +4\varpi^2\Big(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\varpi+8\Big)+\alpha^2\varpi\Big[6\Big(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+2\Big)+19\varpi\Big] \\
&\quad +\alpha^3\Big(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+9\varpi\Big)\Big\} \\
D_T &= 4\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}\Big(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi\Big)^2
\end{aligned} \tag{8.23}$$

The new function $G_T(\varpi)$ maps the interval $[0,\infty]$ of ϖ into the interval $\left[-\frac{3\alpha}{8}, -\frac{3}{8}(4+3\alpha)\right]$ so that the range of the negative variable t is

$$t \in \left[-\frac{3}{8}(4+3\alpha), -\frac{3\alpha}{8}\right] \tag{8.24}$$

and, as expected, the cycle C_1 does not shrink to zero unless $\alpha = 0$. Quite surprisingly the function $G_T(\varpi)$ can be easily inverted and we find:

$$\varpi \equiv H(t,\alpha)^2 = \frac{64t^2 - 9\alpha^2}{54\alpha + 48t + 72} \tag{8.25}$$

while for $P(t)$ we get:

$$P(t,\alpha) = 2\sqrt{\frac{27\alpha^2 + 432\alpha t + 192t(t+3)}{(t+3)(9\alpha+8t+12)\left(9\alpha^2 - 64t^2\right)}} \tag{8.26}$$

and we verify that

$$P(t,0) = \sqrt{\frac{-3}{2t^2+3t}} \tag{8.27}$$

which is the correct result for the singular case $\mathbb{W}P[112]$. In terms of the function $F(t)$ parameterizing the metric (8.17) we have:

$$F(t,\alpha) = \frac{4\left(27\alpha^2 + 432\alpha t + 192t(t+3)\right)}{3(9\alpha+8t+12)\left(64t^2 - 9\alpha^2\right)} = \frac{1}{2}\left(\frac{1}{\frac{3\alpha}{8}+t} - \frac{1}{\frac{3}{8}(3\alpha+4)+t} + \frac{1}{t-\frac{3\alpha}{8}}\right) \tag{8.28}$$

The above structure of the function $F(t,\alpha)$ is very much inspiring. As we see, it is just the sum of three simple poles that are alternatively simple poles of the dt^2-coefficient and zeros of the coefficient of the $(d\tau + (1-\cos(\theta)d\phi)^2$-term. The range of the variable t turns out to be the interval between two such poles where the sign of the function $F(t)$ is the correct one for in order for the metric (8.17) to have Euclidian signature. The three poles are:

$$t_1 = -\frac{3\alpha}{8} \quad ; \quad t_2 = -\frac{3}{8}(3\alpha+4) \quad ; \quad t_3 = \frac{3\alpha}{8} \tag{8.29}$$

We also see what is the mechanism of the degeneration producing the singular $\mathbb{W}P[112]$ case: the two poles t_1 and t_3 come to coincide and the coincidence point is zero. This produces the vanishing of the C_1-cycle as we explained above.

Substituting the function $F(t,\alpha)$ as given in eq. (8.28) into the metric we get a final form of a specific Kähler metric on the second Hirzebruch surface which follows from the Kronheimer construction. This metric provides the boundary condition for the Ricci-flat metric on the canonical bundle tot$K_{\mathbb{F}_2}$ which must reduce to it when setting $ds = 0$, $d\chi = 0$ and $s = -3$.

Verification of the topological conditions for the Kähler metric of $\mathbb{F}_2$. As a matter of check we calculate the periods of the Kähler and Ricci 2-forms also in the real formalism, obtaining the following expected result which holds true for $0 <| \alpha |< 1$:

$$\int_{C_1} \mathbb{K} = \frac{9\alpha}{16} \quad ; \quad \int_{C_2} \mathbb{K} = \frac{9(2+\alpha)}{16} \quad ; \quad \int_{C_1} \mathbb{R}\mathrm{ic} = 0 \quad ; \quad \int_{C_2} \mathbb{R}\mathrm{ic} = 2 \tag{8.30}$$

The above result for the Kähler form is immediate once the function $P(t) = P(t, \alpha)$ is specified. It is instead interesting to see the subtle way in which the result for the Ricci form is obtained independently from the value of α.

Calculating the Ricci tensor of the metric in eq. (8.17) with the function $F(t, \alpha)$ of eq. (8.28) we find the symmetric matrix $\mathcal{R}ic$ which, multiplied by the transpose of the complex structure (8.4) with $P(t)$ as in eq. (8.26) produces the Ricci form $\mathbb{R}\mathrm{ic}_{\mathcal{ED}}$ with the structure displayed in eq. (8.13) and the following explicit expressions for the functions $\mathfrak{A}(t)$ and $\mathfrak{C}(t)$.

$$\mathfrak{A}(t) = \frac{(8t-3\alpha)(3\alpha+8t)\left(27\alpha^2(3\alpha+4)+512t^3+576(3\alpha+4)t^2+216\alpha^2 t\right)}{8t\left(9\alpha^2+144\alpha t+64t(t+3)\right)^2} \tag{8.31}$$

$$\mathfrak{C}(t) = \frac{d}{dt}U(t) \quad ; \quad U(t) = \frac{864(\alpha+1)(\alpha+2)\left(3\alpha^2+8(3\alpha+4)t\right)}{\left(9\alpha^2+144\alpha t+64t(t+3)\right)^2} - \frac{3(3\alpha+4)}{8t} \tag{8.32}$$

We immediately see that $-|t_{max}| = -\frac{3\alpha}{8}$ is a zero of $\mathfrak{A}(t)$ so that $\int_{C1} \mathbb{R}\mathrm{ic}_{\mathcal{ED}} = 0$, while setting as we must $-|t_{min}| = -\frac{3}{8}(3\alpha+4)$ we obtain:

$$U(-|t_{max}) - U(-|t_{min}) = 2 \quad \Rightarrow \quad \int_{C2} \mathbb{R}\mathrm{ic}_{\mathcal{ED}} = 0 \tag{8.33}$$

8.4 The exceptional divisor in symplectic coordinates.

Considering next the description of the 6-dimensional manifold $\mathcal{M}_6$ in terms of symplectic coordinates $\{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \phi, \tau, \chi\}$ (see sect.6) we easily find that the localization $s = -3$ of the exceptional divisor corresponds to $\mathfrak{w} = \frac{3}{2}$, $d\mathfrak{w} = d\chi = 0$. Hence defining

$$\mathcal{D}(\mathfrak{v}) \equiv \mathcal{G}\left(\mathfrak{v}, \frac{3}{2}\right), \tag{8.34}$$

where $\mathcal{G}(\mathfrak{v}, \mathfrak{w})$ is the variable part of the overall symplectic prepotential, we obtain that the Kähler metric on the exceptional divisor has also a description in terms of a symplectic potential given by

$$\mathfrak{D}(\mathfrak{u}, \mathfrak{v}) = G_0(\mathfrak{u}, \mathfrak{v}) + \mathcal{D}(\mathfrak{v}) = \mathcal{D}(\mathfrak{v}) + \left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v} - \mathfrak{u}) + \frac{1}{2}(\mathfrak{u}\log\mathfrak{u} - \mathfrak{u}) - \mathfrak{v}\log\mathfrak{v} \tag{8.35}$$

with moment and angular variables $\mu^i = \{\mathfrak{u}, \mathfrak{v}\}$, $\Theta_j = \{\phi, \tau\}$ and line element as follows:

$$\mathrm{ds}^2_{\mathcal{ED}} = D_{ij} d\mu^i d\mu^j + (D^{-1})^{ij} d\Theta_i d\Theta_j \tag{8.36}$$

where the two matrices are:

$$D_{ij} = \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2 - 2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u}-2\mathfrak{v}} \\ \frac{1}{\mathfrak{u}-2\mathfrak{v}} & \mathcal{D}''(\mathfrak{v}) - \frac{\mathfrak{u}}{\mathfrak{u}\mathfrak{v}-2\mathfrak{v}^2} \end{pmatrix} \quad ; \quad (D^{-1})^{ij} = \begin{pmatrix} \frac{\mathfrak{u}(\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})\mathcal{D}''(\mathfrak{v})+\mathfrak{u})}{\mathfrak{v}^2\mathcal{D}''(\mathfrak{v})} & \frac{\mathfrak{u}}{\mathfrak{v}\mathcal{D}''(\mathfrak{v})} \\ \frac{\mathfrak{u}}{\mathfrak{v}\mathcal{D}''(\mathfrak{v})} & \frac{1}{\mathcal{D}''(\mathfrak{v})} \end{pmatrix} \tag{8.37}$$

Reduced to the exceptional divisor, the coordinate transformation (6.6) is very simple. We have: $\mathfrak{u} = \frac{3}{4}t(-1+\cos\theta)$, $\mathfrak{v} = -\frac{3t}{4}$. So if we declare that the function $\mathcal{D}(\mathfrak{v}) = \Pi(t)$, is a function of t we obtain

$\mathcal{D}''(\mathfrak{v}) = \frac{16}{9}\Pi''(t)$ and replacing these transformation in (8.34-8.35) we obtain that the line element in symplectic coordinates coincides with the line element of eq. (8.2) provided that:

$$\mathcal{D}(\mathfrak{v}) \equiv \Pi(t) \quad ; \quad P(t)^2 = -\frac{4\Pi''(t)}{t+3} \quad \Rightarrow \quad \Pi''(t) = -\frac{3}{4}F(t) \tag{8.38}$$

So the function $F(t)$ determining the Kähler geometry of the exceptional divisor, linked to its Kähler potential by eq. (8.19), is just $4/3\times$ the second derivative of the non-fixed part of the symplectic potential.

The case of the Kähler metric on $\mathbb{F}_2$ with generic α. Applying the above scheme to the Kähler metric on $\mathbb{F}_2$ induced by the Kronheimer construction, namely utilizing in eq. (8.38) $F(t) = F(t,\alpha)$ as given in eq. (8.28) we obtain the following differential equation:

$$\mathcal{D}''(\mathfrak{v};\alpha) = 16\left(\frac{1}{27\alpha - 32\mathfrak{v} + 36} + \frac{1}{32\mathfrak{v} - 9\alpha} + \frac{1}{9\alpha + 32\mathfrak{v}}\right) \tag{8.39}$$

which, modulo linear functions implies $\mathcal{D}(\mathfrak{v},\alpha) = \mathcal{D}_{\mathbb{F}_2}(\mathfrak{v},\alpha)$ where

$$\begin{aligned}\mathcal{D}_{\mathbb{F}_2}(\mathfrak{v},\alpha) \equiv & \frac{1}{2}\mathfrak{v}\log\left(1024\mathfrak{v}^2 - 81\alpha^2\right) + \frac{1}{64}(27\alpha - 32\mathfrak{v} + 36)\log(27\alpha - 32\mathfrak{v} + 36) \\ & + \frac{9}{32}\alpha\,\mathrm{arctanh}\left(\frac{32\mathfrak{v}}{9\alpha}\right) - \frac{\mathfrak{v}}{2}\end{aligned} \tag{8.40}$$

We also find:

$$\mathcal{D}_{\mathbb{F}_2}(\mathfrak{v},0) = -\frac{1}{16}(-9 + 8\mathfrak{v})\log\left(3 - \frac{8}{3}\mathfrak{v}\right) + \mathfrak{v}\log\mathfrak{v} \quad \text{modulo a linear function of } \mathfrak{v} \tag{8.41}$$

For $\alpha = 0$ this is the correct result for $\mathbb{W}P[112]$.

9 The Monge-Ampère equation and its series expansion

In this section we arrive at the core of the issue, *i.e.* the construction of Ricci-flat metrics on the spaces we are concerned with. The common general feature of these is that they are the total space of the canonical bundle of a complex two-dimensional compact Kähler manifold $\mathcal{M}_4$, the *exceptional divisor* when the total space is the full or partial resolution of a quotient singularity. In this interpretation the base of the canonical bundle is indeed the exceptional divisor produced by the *blow up* of an isolated singular point.

The additional common structural feature of the Ricci-flat metrics we want to consider is, as we already stressed several times, the group of continuous isometries that they should possess, mentioned in equation (1.8). The action of G_{iso} on the three complex coordinates u, v, w that originate from the integration of the complex structure was displayed in eq.s (7.1). The presence of these isometries imposes very stringent constraints on the Kähler metric that are most efficiently handled at the level of the potential $\mathfrak{P}$ from which the metric can be obtained by means of derivatives. The condition of Ricci-flatness of the metric is translated into a nonlinear differential equation to be satisfied by the potential $\mathfrak{P}$ that we name the Monge-Ampère equation.

As we have seen in the previous pages, there are three equivalent formulations of the Kähler geometry of the toric six-dimensional manifolds $\mathcal{M}_6$ we are concerned with:

A) The complex setup where the geometry is encoded in the Kähler potential $\mathfrak{P} = \mathcal{K}(u, v, w, \overline{u}, \overline{v}, \overline{w})$

B) The symplectic setup where the geometry is encoded in the symplectic potential $\mathfrak{P} = \mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$

C) The hybrid setup where the geometry is encoded in the symplectic potential, but instead of the coordinates v, w we use the coordinates s,t related to them by the coordinate transformation (6.6-6.7).

Correspondingly there are, to begin with, two formulations of the Monge-Ampère equation, one for the Kähler potential, one for the symplectic potential. In both cases the constraints imposed by the chosen isometries reduce the effective potential to be a function of only two real variables so that the Monge-Ampère equation is a non linear partial differential equation in two variables. At this point the symplectic case still splits into two versions depending on whether we employ the pure symplectic variables or the hybrid ones s, t.

In all formulations, as we show below, the equation has the property that we can fix as boundary condition an arbitrarily chosen Kähler metric on the exceptional divisor.

9.1 The Monge-Ampère equation for the Kähler potential

We begin with the Monge-Ampère equation written in terms of the Kähler potential. It follows from the chosen isometries that the Kähler potential $\mathcal{K}$ must be a function only of the two invariants:

$$\mathfrak{f} \equiv \mid w \mid^2 \quad ; \quad \varpi \equiv \left(1+ \mid u \mid^2\right)^2 \mid v \mid^2 \quad \text{or} \quad \mathcal{T} \equiv 4 + \varpi - \sqrt{\varpi\,(\varpi + 8)} \tag{9.1}$$

so that we can set:

$$\mathcal{K} = G(\varpi, \mathfrak{f}) \quad \text{or} \quad \mathcal{K} = G(\mathcal{T}, \mathfrak{f}) \tag{9.2}$$

The use of the alternative combination $\mathcal{T}$ simplifies the Kähler potential in certain cases.

The Monge-Ampère equation in this setup is simply the statement that the determinant of the Kähler metric is constant. Indeed in the complex coordinate setup the hermitian Ricci tensor is obtained from the logarithm of the metric determinant in the same way as the Kähler metric is obtained from the Kähler potential:

$$\mathrm{Ric}_{ij^\star} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \overline{z}^{j^\star}} \log\left[\det \mathbf{g}\right] \quad ; \quad \mathbf{g} = g_{ij^\star} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \overline{z}^{j^\star}} G(\mathcal{T}, \mathfrak{f}) \quad ; \quad z^i \equiv \{u, v, w\} \tag{9.3}$$

Hence if

$$\det \mathbf{g} = \kappa \tag{9.4}$$

where κ is a constant parameter, the Ricci tensor is necessarily zero and we have a Ricci-flat metric. The Monge-Ampère equation is obtained by replacing in eq. (9.4) the expression of $\det \mathbf{g}$ in terms of derivatives of the Kähler potential $G(\mathcal{T}, \mathfrak{f})$. Relying on the definition of the invariants provided in eq. (9.1) we obtain:

$$\begin{aligned} 4\mathcal{T}^3 G^{(1,0)}(\mathcal{T},\mathfrak{f}) \Big\{ & G^{(0,1)}(\mathcal{T},\mathfrak{f}) \left[\left(\mathcal{T}^2 + 8\mathcal{T} - 16\right) G^{(1,0)}(\mathcal{T},\mathfrak{f}) + \mathcal{T}\left(\mathcal{T}^2 - 16\right) G^{(2,0)}(\mathcal{T},\mathfrak{f})\right] \\ & + \mathfrak{f}\left[G^{(0,2)}(\mathcal{T},\mathfrak{f})\left\{\left(\mathcal{T}^2 + 8\mathcal{T} - 16\right) G^{(1,0)}(\mathcal{T},\mathfrak{f}) + \mathcal{T}\left(\mathcal{T}^2 - 16\right) G^{(2,0)}(\mathcal{T},\mathfrak{f})\right\}\right. \\ & \left.\left. - \mathcal{T}\left(\mathcal{T}^2 - 16\right) G^{(1,1)}(\mathcal{T},\mathfrak{f})^2\right]\right\} = -\kappa(\mathcal{T}+4)^4 \end{aligned} \tag{9.5}$$

One can solve the Monge-Ampère equation in the above form by developing the Kähler potential in power series of $\mathfrak{f}$:

$$G(\mathcal{T}, \mathfrak{f}) = G_0(\mathcal{T}) + \sum_{n=1}^{\infty} G_n(\mathcal{T})\, \mathfrak{f}^n \tag{9.6}$$

where $G_0(\mathcal{T})$ is the Kähler potential of a convenient Kähler metric defined over the exceptional divisor.

Indeed it is a property of the considered system that inserting (9.6) into the Monge-Ampère equation (9.5), the function $G_0(\mathcal{T})$ corresponding to the Kähler potential of the Kähler metric on the exceptional divisor is undetermined, while all the other $G_n(\mathcal{T})$ functions can be iteratively determined in terms of the previous $G_{k<n}(\mathcal{T})$.

As we discussed before, it is quite remarkable that on the exceptional divisor located at $s = -3$ the Ricci-flat orthotoric metric (5.9) reduces precisely to the Kähler metric on $\mathbb{W}P[112]$, which was obtained in [7] from the Kronheimer construction while performing the partial resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity on a type III wall.

9.1.1 Recursive solution for the Kähler potential in the case $\mathrm{tot}K_{\mathbb{W}P[112]}$

In this section we present the recursive solution of the Monge-Ampère equation which was obtained by means of a dedicated MATHEMATICA code using as 0-th order Kähler potential the following one

$$G_0(\mathcal{T}) = 4\log\mathcal{T} + \mathcal{T} \tag{9.7}$$

which yields the Kronheimer Kähler metric on $\mathbb{W}P[112]$.

The Kähler potential of the full metric on $\mathrm{tot}K_{\mathbb{W}P[112]}$ can be expressed as follows

$$G(\mathcal{T},\mathfrak{f}) = 4\log\mathcal{T} + \mathcal{T} + \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{(k!)}\frac{\mathcal{P}_{2k-2}(\mathcal{T})}{(2\mathcal{T})^{2k}(\mathcal{T}+4)^{2k-3}}(\kappa\mathfrak{f})^k\right) \tag{9.8}$$

where the symbol $\mathcal{P}_{2k-2}(\mathcal{T})$ denotes a polynomyal of degree $2k-2$ in the variable $\mathcal{T}$. The remarkable feature is that the coefficients of the polynomials $\mathcal{P}_{2k-2}(\mathcal{T})$ are all integer numbers whose decomposition into prime factors involves prime numbers of increasing values. We show the first 6 of these intriguing polynomials

$k = 1 \mid \mathcal{P}_0(\mathcal{T}) = 2$

$k = 2 \mid \mathcal{P}_2(\mathcal{T}) = 112 + 16\mathcal{T} + \mathcal{T}^2$

$k = 3 \mid \mathcal{P}_4(\mathcal{T}) = 2\left(10112 + 4000\mathcal{T} + 408\mathcal{T}^2 + 30\mathcal{T}^3 + \mathcal{T}^4\right)$

$k = 4 \mid \mathcal{P}_6(\mathcal{T}) = 6563840 + 43347392\mathcal{T} + 925952\mathcal{T}^2 + 82624\mathcal{T}^3 + 7112\mathcal{T}^4 + 350\mathcal{T}^5 + 8\mathcal{T}^6$

$k = 5 \mid \mathcal{P}_8(\mathcal{T}) = 3128950784 + 2919825408\mathcal{T} + 987267072\mathcal{T}^2 + 150301696\mathcal{T}^3 + 13354240\mathcal{T}^4 + 1313920\mathcal{T}^5 + 76064\mathcal{T}^6 + 2812\mathcal{T}^7 + 49\mathcal{T}^8$

$k = 6 \mid \mathcal{P}_{10}(\mathcal{T}) = 1980772122624 + 2387983728640\mathcal{T} + 1118459035648\mathcal{T}^2 + 256754671616\mathcal{T}^3 + 32204621824\mathcal{T}^4 + 3128804864\mathcal{T}^5 + 331169920\mathcal{T}^6 + 20666912\mathcal{T}^7 + 975904\mathcal{T}^8 + 28886\mathcal{T}^9 + 407\mathcal{T}^{10}$

9.2 The Monge-Ampère equation of Ricci-flatness for the symplectic potential

According to [60, 61] the condition for Ricci-flatness can be written as a differential condition on the symplectic potential which is the following

$$\mathrm{Det}\left[G_{ij}\right] = \mathrm{const} \times \mathrm{Exp}\sum_{h=1}^{n} c^h \partial_h G \tag{9.9}$$

where c^h are some constants. In the case of our general metric with isometry $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)$, the symplectic form of the Monge-Ampère equation simplifies since we have the particular form (6.13) of the matrix G_{ij}. Indeed we find:

$$\text{detHes} \equiv \text{Det}\left[G_{ij}\right] = \frac{\mathfrak{v}}{\mathfrak{u}(\mathfrak{u}-2\mathfrak{v})}\left[\mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w})^2 - \mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w})\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w})\right] \tag{9.10}$$

This facilitates the study of the Ricci-flatness condition because the coefficients $c^{\mathfrak{u}}$ and $c^{\mathfrak{v}}$ are already fixed by the need to reproduce the $\mathfrak{u}$-dependence of detHes. We easily find:

$$\text{Exp}\left[-2\partial_{\mathfrak{u}}G(\mathfrak{u},\mathfrak{v},\mathfrak{w}) \; - 2\partial_{\mathfrak{v}}G(\mathfrak{u},\mathfrak{v},\mathfrak{w}) \; + \; k\,\partial_{\mathfrak{w}}G(\mathfrak{u},\mathfrak{v},\mathfrak{w})\right] \; = \; -\frac{e^{k\mathcal{G}^{(0,1)}(\mathfrak{v},\mathfrak{w})-2\mathcal{G}^{(1,0)}(\mathfrak{v},\mathfrak{w})}\mathfrak{v}^2}{\mathfrak{u}^2-2\mathfrak{u}\mathfrak{v}} \tag{9.11}$$

Hence in the symplectic formalism the Monge-Ampère equation for Ricci flatness reduces to the following relation:

$$c e^{k\mathcal{G}^{(0,1)}(\mathfrak{v},\mathfrak{w})-2\mathcal{G}^{(1,0)}(\mathfrak{v},\mathfrak{w})}\mathfrak{v} + \mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w})^2 - \mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w})\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w}) = 0 \tag{9.12}$$

imposed solely on the function of two variables $\mathcal{G}[\mathfrak{v},\mathfrak{w}]$.

We have explicitly verified that the function $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w})$ defined in equation (6.18), which corresponds to the orthotoric Ricci-flat metric on $\text{tot}K_{\mathbb{W}P[112]}$ satisfies eq. (9.12) with:

$$k = -\frac{8}{3} \quad ; \quad c = \frac{72e^3}{7} \tag{9.13}$$

9.2.1 Discussion of the boundary condition

As we show below, differently from the case of the Monge-Ampère equation for the Kähler potential in the symplectic case, there is a subtle issue concerning the choice of boundary condition to be imposed on the function while restricting it to the exceptional divisor. The important point is that at the level of the metric the limit $\mathfrak{w} \to \frac{3}{2}$ should reproduce the metric on the divisor derived from the potential $\mathcal{D}(\mathfrak{v}) = \mathcal{G}\left(\mathfrak{v},\frac{3}{2}\right)$. There are only two ways to obtain this. If the symplectic potential $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ is holomorphic at $\mathfrak{w} = \frac{3}{2}$ and admits a Taylor series expansion in $\mathfrak{w}-\frac{3}{2}$ we are obliged to impose that $\partial_w\mathcal{G}(\mathfrak{v},\mathfrak{w})$ be a constant at $\mathfrak{w} = \frac{3}{2}$ and this results in a recursive solution with coefficients that are rational functions of increasing order and can hardly define a convergent series. Furthermore the only known solution of the Monge Ampère equation, provided by the function (6.18) corresponding to the orthotoric metric on $\text{tot}K_{\mathbb{W}P[112]}$ has not this holomorphic behavior. Indeed $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w})$ provides a paradigmatic example of the other possible boundary condition which foresees a logarithmic singularity of the symplectic potential while approaching the exceptional divisor:

$$\mathcal{G}(\mathfrak{v},\mathfrak{w}) \overset{\mathfrak{w}\to\frac{3}{2}}{\approx} \left(\mathfrak{w}-\tfrac{3}{2}\right)\log\left(\mathfrak{w}-\tfrac{3}{2}\right) + \mathcal{G}_0(\mathfrak{v}) + \mathcal{O}\left(\mathfrak{w}-\tfrac{3}{2}\right) \tag{9.14}$$

In the sequel we show that with the second type of boundary condition we can reconstruct the known solution $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w})$ of equation (6.18) and also derive a series solution pertaining to the smooth $\mathbb{F}_2$ case which displays the same general features as $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w})$. Unfortunately, up to the present moment we can only give numerical evidences of the last statement.

In view of what we explained above we skip the details concerning the first type of boundary condition (holomorphicity at $\mathfrak{w} = \frac{3}{2}$ and jump directly to the case of a logarithmic singularity at $\mathfrak{w} = \frac{3}{2}$. Indeed, a logarithmic singularity is known to be the correct behaviour to ensures smoothness of the toric Kähler metrics near to divisors [62, 63, 60].

9.3 The boundary condition with a logarithmic singularity at $\mathfrak{w} = \frac{3}{2}$

We implement the second type of boundary condition requiring that following two properties should be preserved:

a) The symplectic potential $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ has a finite limit for $\mathfrak{w}\to\frac{3}{2}$

b) The limit for $\mathfrak{w}\to\frac{3}{2}$ of the bundle metric should be exactly the exceptional divisor metric (8.36-8.37)

Namely we must have:

$$\lim_{\mathfrak{w}\to\frac{3}{2}}\mathcal{G}(\mathfrak{v},\mathfrak{w}) \;=\; \mathcal{D}(\mathfrak{v}) \quad ; \quad \lim_{\mathfrak{w}\to\frac{3}{2}}\mathrm{ds}^2_{symp} \;=\; \mathrm{ds}^2_{\mathcal{ED}} \tag{9.15}$$

To discuss this alternative boundary condition it is convenient to use rescaled variables defined as follows

$$x=2\mathfrak{v} \quad ; \quad y=3\mathfrak{w} \quad ; \quad y=\frac{9}{2}+\omega \Rightarrow \omega=3\left(\mathfrak{w}-\tfrac{3}{2}\right) \tag{9.16}$$

In terms of such variables the Monge-Ampère equation (9.12) becomes

$$c\,x\,\exp\Big[-8\,\mathcal{G}^{(0,1)}(x,\omega)-4\,\mathcal{G}^{(1,0)}(x,\omega)\Big]-\mathcal{G}^{(1,1)}(x,\omega)^2+\mathcal{G}^{(0,2)}(x,\omega)\mathcal{G}^{(2,0)}(x,\omega)\;=\;0 \tag{9.17}$$

Instead of assuming that $\mathcal{G}(x,\omega)$ is holomorphic at $\omega=0$, we impose that it has a logarithmic singularity of the form $\omega\log\omega$. Indeed this is the unique alternative way in which the metric on the total space can reduce to the metric exceptional divisor in the limit $\omega\to 0$. Furthermore this behavior for $\mathfrak{w}\to\frac{3}{2}$ is precisely that displayed by the symplectic potential $\mathcal{G}_{\mathbb{W}P[112]}(\mathfrak{v},\mathfrak{w})$ explicitly written down in eq. (6.18). Hence we assume the following different series expansion which isolates a logarithmic singularity at $\omega=0$:

$$\mathcal{G}(x,\omega)\;=\;\tfrac{1}{8}\,\omega\,\log[\omega]+\mathcal{G}_0(x)\;+\;\sum_{k=1}^{\infty}\omega^k\mathcal{G}_k(x) \tag{9.18}$$

The function $\mathcal{G}_0(x)$ is free. All the functions $\mathcal{G}_k(x)$ ($k\geq$1) are determined in terms of $\mathcal{G}_0(x)$. For instance we have:

$$\begin{aligned}
\mathcal{G}_0(x) &= \mathcal{G}_0(x)\\
\mathcal{G}_1(x) &= \frac{1}{8}\log\left(-\frac{e^{-4\mathcal{G}_0'(x)}x}{\mathcal{G}_0''(x)}\right)\\
\mathcal{G}_2(x) &= \frac{32x^2\mathcal{G}_0''(x)^3-12x\mathcal{G}_0''(x)^2+16x^2\mathcal{G}_0{}^{(3)}(x)\mathcal{G}_0''(x)+2\mathcal{G}_0''(x)-2x\mathcal{G}_0{}^{(3)}(x)+x^2\mathcal{G}_0{}^{(4)}(x)}{256x^2\mathcal{G}_0''(x)^2}\\
\mathcal{G}_3(x) &= \frac{1}{36864x^4\mathcal{G}_0''(x)^4}\Big(-48x^2\left(16\mathcal{G}_0{}^{(3)}(x)x^2+7\right)\mathcal{G}_0''(x)^4-48x\left(12x^3\mathcal{G}_0{}^{(4)}(x)-5\right)\mathcal{G}_0''(x)^3\\
&\quad +4\left(144\mathcal{G}_0{}^{(3)}(x)^2x^4-18\mathcal{G}_0{}^{(5)}(x)x^4+38\mathcal{G}_0{}^{(4)}(x)x^3+12\mathcal{G}_0{}^{(3)}(x)x^2-11\right)\mathcal{G}_0''(x)^2\\
&\quad +2x\Big(-152x^2\mathcal{G}_0{}^{(3)}(x)^2+\left(72\mathcal{G}_0{}^{(4)}(x)x^3+4\right)\mathcal{G}_0{}^{(3)}(x)\\
&\quad -x\left(\mathcal{G}_0{}^{(6)}(x)x^2-6\mathcal{G}_0{}^{(5)}(x)x+2\mathcal{G}_0{}^{(4)}(x)\right)\Big)\mathcal{G}_0''(x)+9x^2\left(x\mathcal{G}_0{}^{(4)}(x)-2\mathcal{G}_0{}^{(3)}(x)\right)^2\Big)
\end{aligned} \tag{9.19}$$

9.3.1 Recursive solution of the symplectic Monge-Ampère equation in the case where the smooth $\mathbb{F}_2$ surface is at the boundary

Relying on the results of the previous subsection we consider the case where the symplectic potential at the boundary (*i.e.* on the exceptional divisor) is the one yielding the Kronheimer Kähler metric on $\mathbb{F}_2$. In terms of the x variable and setting, $\Delta=\frac{9}{8}\alpha$ the function in eq. (8.40) can be rewritten as follows:

$$\mathcal{G}_0(x,\Delta)\equiv\frac{1}{4}x\log\left(4x^2-\Delta^2\right)+\frac{1}{8}\,\Delta\log\left(\frac{\Delta+2x}{\Delta-2x}\right)+\frac{1}{8}\left(3\Delta-2x+\frac{9}{2}\right)\log\left(3\Delta-2x+\frac{9}{2}\right) \tag{9.20}$$

from the formal solution discussed in the previous section we obtain:

$$\mathcal{G}(x,\omega) = \mathcal{G}_0(x,\Delta) + \frac{1}{8}\omega \log\left(\frac{x(6\Delta-4x+9)^2}{2e\left[\Delta^2+4x^2-6(2\Delta+3)x\right]}\right) + \frac{1}{8}\,\omega\,\log\omega + \sum_{k=1}^{\infty}\frac{N_{k+1}(x,\Delta)}{D_{k+1}(x,\Delta)}\,\omega^{k+1} \quad (9.21)$$

where $N_{k+1}(x,\Delta)$ and $D_{k+1}(x,\Delta)$ are polynomials whose degrees are as follows:

$$\text{degree}\left[N_{k+1}(x,\Delta)\right] \;=\; 6\,k \quad ; \quad \text{degree}\left[D_{k+1}(x,\Delta)\right] \;=\; 7\,k \quad (9.22)$$

Hence the degree of the coefficient of ω^{k+1} in the series expansion is a rational function of x of degree $-k$, a feature that looks promising for convergence.

By means of a dedicated MATHEMATICA code we can calculate the polynomials $N_{k+1}(x,\Delta)$, $D_{k+1}(x,\Delta)$ to any desired order. For reason of typographical space we display here only the first terms up to order $k=2$.

$N_2(\mathrm{x}) = \,-3888x^4-864x^5+128x^6-5184x^4\Delta-576x^5\Delta+720x^3\Delta^2-1536x^4\Delta^2+480x^3\Delta^3-81\Delta^4+162x\Delta^4-120x^2\Delta^4-108\Delta^5+108x\Delta^5-36\Delta^6$

$N_3(\mathrm{x}) = 49152x^{12}-811008x^{11}(3+2\Delta)-891\Delta^8(3+2\Delta)^4+108x\Delta^6(3+2\Delta)^3\left(810+1080\Delta+421\Delta^2\right)-36x^2\Delta^6(3+2\Delta)^2\left(23652+31536\Delta+10961\Delta^2\right)+384x^6\Delta^4\left(31347+41796\Delta+12692\Delta^2\right)$
$+1024x^{10}\left(42039+56052\Delta+18412\Delta^2\right)-1536x^7\Delta^2\left(-9963-19926\Delta-8238\Delta^2+412\Delta^3\right)$
$-1536x^5\Delta^4\left(15795+31590\Delta+19092\Delta^2+3368\Delta^3\right)$
$-3072x^9\left(28431+56862\Delta+41124\Delta^2+10568\Delta^3\right)+576x^4\Delta^4(35721+95256\Delta$
$+83151\Delta^2+26196\Delta^3+1655\Delta^4)$
$+256x^8\left(-137781-367416\Delta-270540\Delta^2-34128\Delta^3+23272\Delta^4\right)$
$+864x^3\Delta^4\left(-6561-21870\Delta-17739\Delta^2+3402\Delta^3+8805\Delta^4+2558\Delta^5\right)$

$D_2(\mathrm{x}) = 128x^2(-9+4x-6\Delta)\left(-18x+4x^2-12x\Delta+\Delta^2\right)^2$

$D_3(\mathrm{x}) = 9216x^4(9-4x+6\Delta)^2\left(4x^2+\Delta^2-6x(3+2\Delta)\right)^4$

9.3.2 Numerical study in the case $\Delta = \frac{3}{4}$

Since so far we have not been able to guess the sum of the series in terms of elementary or higher transcendental functions, to get some understanding of the solution we have resorted to a numerical study of the approximants to the solution obtained by truncating the series in eq. (9.21) to various orders performing the plots.

The relevant thing is that for the special value $\Delta = 0$ of the parameter we know the exact sum of the series. It is provided by the symplectic potential (6.18) which pertains to the case of $\text{tot}K_{\mathbb{W}P[112]}$. This fortunate occurrence enables us to compare the plot of the exact function with those of its approximants. This comparison, as we are going to see, turns out to be quite inspiring since it elucidates the meaning of certain oscillatory behaviors of the approximants that are completely analogous in the case $\Delta = 0$, where we know the sum of the series and in the case $\Delta > 0$ where the sum is unknown.

In terms of the variables x and ω the symplectic potential of the orthotoric metric takes the following explicit expression:

$$
\begin{aligned}
\mathcal{G}_{\Delta=0}(x,\omega) \;=\; & \frac{1}{224}\left\{7\left[(4\omega+16x)\log\left(\frac{9}{2}-\sqrt{\left(x+\omega+\frac{9}{2}\right)^2-18x}-x-\omega\right)\right.\right.\\
& -2(4x-4\omega-9)\log\left(\sqrt{\left(x+\omega+\frac{9}{2}\right)^2-18x}-x-\omega\right)\\
& \left.+(4x+2\omega+9)\log\left(\frac{1}{567}\left[4\sqrt{\left(x+\omega+\frac{9}{2}\right)^2-18x}+4x+4\omega+27\right]^2+1\right)\right]\\
& -2\sqrt{7}(4x-2\omega-27)\arctan\frac{4\sqrt{\left(x+\omega+\frac{9}{2}\right)^2-18x}+4x+4\omega+27}{9\sqrt{7}}\\
& \left.-(4x+9)\log\frac{34359738368}{823543}+2\sqrt{7}(4x-27)\arctan\frac{5}{\sqrt{7}}\right\} \qquad (9.23)
\end{aligned}
$$

For comparison we choose the series solution in the case $\Delta=\frac{3}{4}$. This value, corresponding to $\alpha=\frac{2}{3}$, introduces various simplifications in the solution and, for no other good reason, provides a good reference point.

In this case the symplectic potential takes the following appearance

$$
\begin{aligned}
G_{\Delta=\frac{3}{4}}(x,\omega) \;=\; & \frac{1}{32}\left[8x\log\left(4x^2-\frac{9}{16}\right)+(27-8x)\log\left(\frac{27}{4}-2x\right)+6\,\mathrm{arctanh}\frac{8x}{3}\right]\\
& \frac{1}{8}\omega\log(\omega)+\frac{1}{8}\omega\log\left(\frac{2(27-8x)^2x}{e[16x(4x-27)+9]}\right)+\sum_{k=1}^{\infty}\frac{\widehat{N}_{k+1}(x)}{\widehat{D}_{k+1}(x)}\,\omega^{k+1} \qquad (9.24)
\end{aligned}
$$

We omit the explicit presentation of the rational functions $\frac{\widehat{N}_{k+1}(x)}{\widehat{D}_{k+1}(x)}$ that we have calculated by means of a computer programme up to order $k=10$ and higher. We rather present the plots of such approximants. Let us first consider the plot of the function $\mathcal{G}_{\Delta=0}(x,\omega)$ displayed in fig.6.

As we distinctly see from the picture, the exact function, namely the sum of the infinite series in ω defines parametrically a perfectly smooth surface in three dimensions that however features a nontrivial structure provided by a sort of *smooth bending* along a line that starts approximately at $x=\frac{9}{4},\omega=0$ and goes up towards $x=\frac{9}{2},\omega=\infty$.

The geometrically meaning of this bending is not entirely clear, yet one can guess that it corresponds to a transition region from a *near divisor geometry* to an asymptotic geometry that is that of a metric cone over the Sasakian orbifold $\mathbb{S}^5/\mathbb{Z}_4$.

It is now interesting to compare the behavior of the exact function with its approximants obtained truncating the series to various orders. Let us now consider the plots displayed in fig.7.

In the plot on the right, the surface plotted in the middle is the sum of the series (*i.e.* the exact function), while the other two surfaces, respectively bending, one up, the other down, are two consecutive approximants (the first of even order, the second of odd order). As we clearly see, the series converges to the exact function and does it rapidly, in the region before the bending structure illustrated above. As we come close to such a line of bending the series no longer converges and its various truncations oscillate violently creating a peculiar canyon.

Let us now compare this behavior of the case $\Delta=0$ with that of the series solution for $\Delta=\frac{3}{4}$. To this effect let us consider the figure 8. The structure of the plots of the truncated series are qualitatively the same in the case $\Delta=\frac{3}{4}$, as they are in the case $\Delta=0$. Furthermore, in a completely analogous way to the case $\Delta=0$, for small values of ω and x also the series representation of $G_{\Delta=\frac{3}{4}}(x,\omega)$ converges

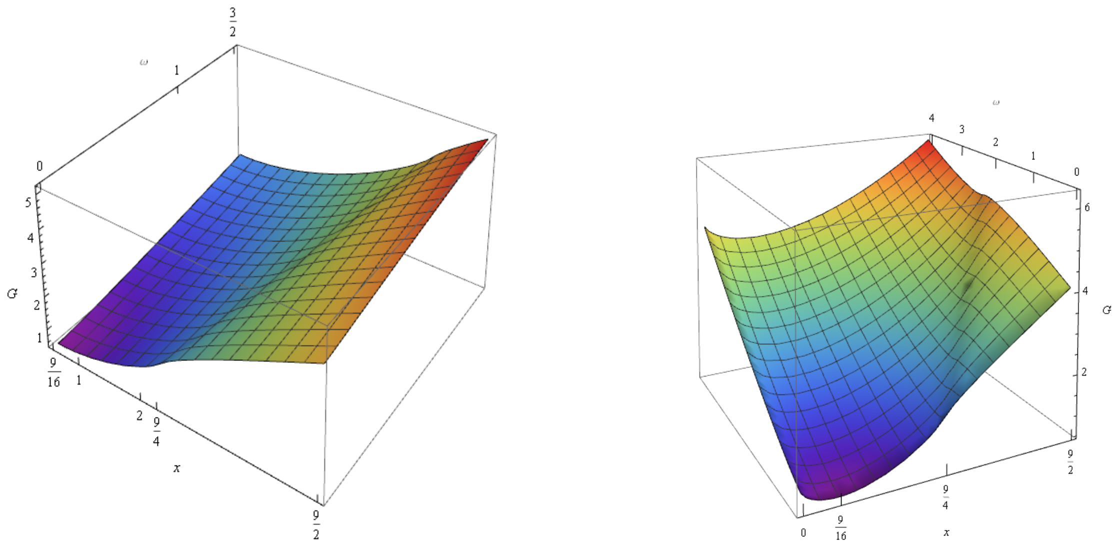

Figure 6: Plots of the exact symplectic potential $\mathcal{G}_{\Delta=0}(x,\omega)$ for small values of the distance ω from the exceptional divisor (plot on the left) and extending to large values (plot on the right.)

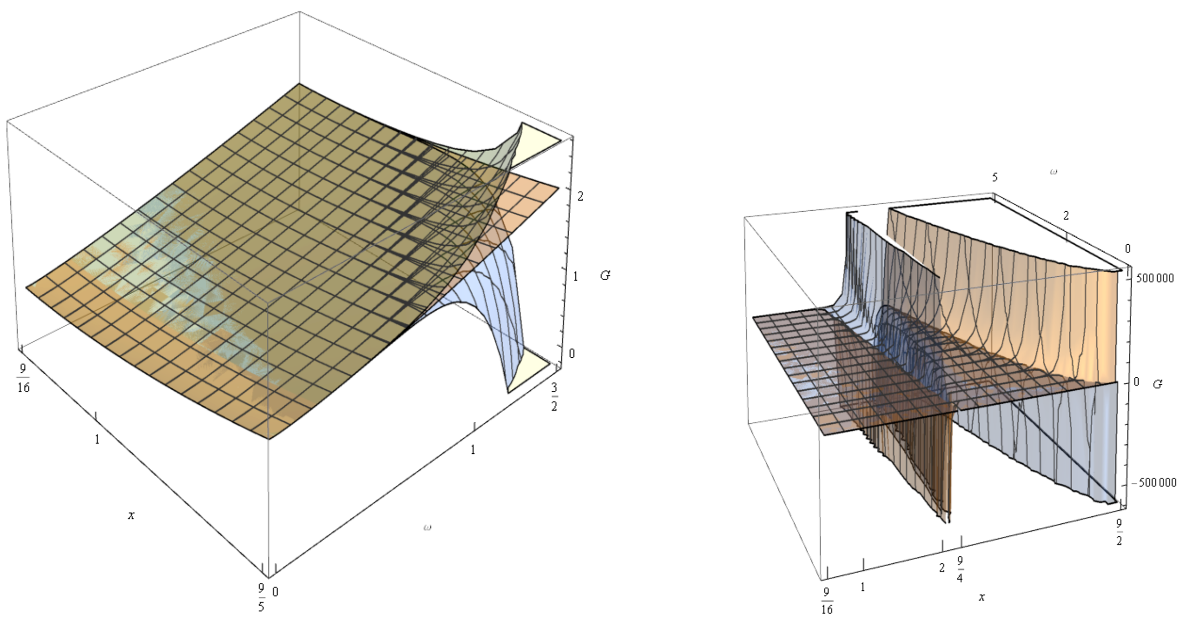

Figure 7: Plots of the exact symplectic potential $\mathcal{G}_{\Delta=0}(x,\omega)$ compared to its approximants of order 6 and 7 respectively: on the right for small values of ω, on the left extending to large values of ω.

rapidly to some well defined function while approaching the region of the bending it starts oscillating. Hence we are led to conclude that we should be able to retrieve an analytically defined solution of the Monge-Ampère equation for the symplectic potential which reduces to the Kronheimer metric on $\mathbb{F}_2$ at $\omega = 0$. It is a matter of finding some alternative way of summing the series by a smart change of variables or by means of some smart integral transform.

9.4 The Hybrid version of the Monge-Ampère equation

The most promising setup to study the MA equation for the symplectic potential is the hybrid one. Working in the s,t coordinates defined in eq.s (6.6-6.7) and setting $\mathcal{G}(v,w) = \Gamma(t,s)$, the equation (9.12)

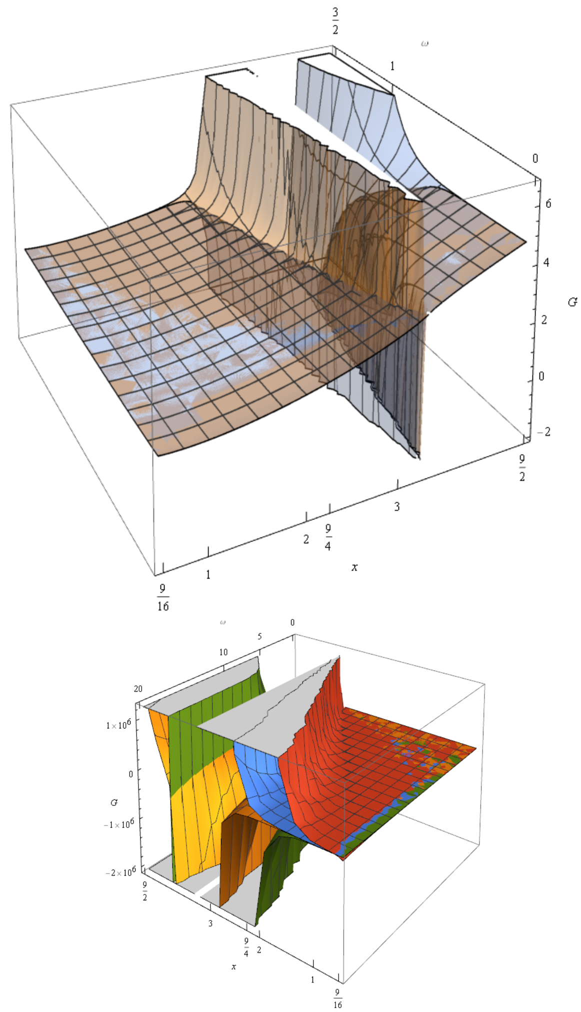

Figure 8: Plots of approximants of even and odd order of the function $\mathcal{G}_{\Delta=\frac{3}{4}}(x,\omega)$ the sum of whose series representation is unknown. As in the other cases the plot on the right is for small values of ω and displays two consecutive approximants of order 7 and 8, respectively, while the plot on the left extends to large values of ω and displays several approximants.

is transformed into the following one:

$$\frac{1}{4}ce^{\mathcal{C}}st = \frac{64\mathcal{B}}{(s-t)^2} - \frac{64\mathcal{A}}{(s-t)^4} \tag{9.25}$$

where:

$$\begin{aligned}
\mathcal{C} &\equiv \frac{8(s+1)\Gamma^{(0,1)}(t,s) - 8(t+1)\Gamma^{(1,0)}(t,s)}{s-t} \\
\mathcal{B} &\equiv \Gamma^{(0,2)}(t,s)\Gamma^{(2,0)}(t,s) \\
\mathcal{A} &\equiv \left(\Gamma^{(0,1)}(t,s) - \Gamma^{(1,0)}(t,s) + (s-t)\Gamma^{(1,1)}(t,s)\right)^2
\end{aligned} \tag{9.26}$$

It is an important observation that the term $\mathcal{A}$ is the square of the constraint whose vanishing implies the orthotoric separation of the s,t variables (see the last of eq.s (6.15). It is interesting to see how with this separation of variables, namely when $\mathcal{A}=0$, the differential equation (9.25) does indeed split in two equations, one for the t variable, the other for the s variable. On the other hand the equation for the t variable implies the $\mathbb{W}P[112]$ symplectic potential.

The argument goes as follows. Generalizing the structure of the known solution for the case $\mathbb{W}P[112]$ we introduce the following ansatz:

$$\Gamma(s,t) \;=\; -\frac{1}{3}(2s+3)\Pi(t)+\mathcal{P}(s)+(s+3)\left(\mathcal{Q}(t)-\frac{1}{16}(t+3)\log(s+3)\right)+st\mathrm{Y}_1(s)+(s+t)\mathrm{Y}_2(s) \quad (9.27)$$

where $\Pi(t)$ is an unknown function of t that we would like to identify with the symplectic potential of the exceptional divisor metric and $Y_{1,2}(s)$ are also two unknown functions of s. On the other hand the other two functions entering the ansatz are integral differential functionals of $Y_{1,2}(s)$ and $\Pi(t)$, respectively :

$$\begin{aligned} \mathcal{P}(s) &= \frac{1}{16}\int \left(-16\kappa_1 - 16s^2\mathrm{Y}_1'(s) - 32s\mathrm{Y}_2'(s) + s + 3\right) ds & (9.28)\\ \mathcal{Q}(t) &= \frac{1}{3}(t+3)\int \frac{(2t+3)\Pi'(t)-2\Pi(t)}{(t+3)^2}\, dt + \kappa_1 & (9.29)\end{aligned}$$

With these choices the term $\mathcal{C}$ in eq. (9.26) splits into separate functions of different variables:

$$\begin{aligned} \mathcal{C} &= \mathcal{U}(s)+T(t)\\ T(t) &= \frac{8\left(3(t+1)\Pi'(t)+2(t+3)\int \frac{(2t+3)\Pi'(t)-2\Pi(t)}{(t+3)^2}\, dt - 4\Pi(t)\right)}{3(t+3)}\\ \mathcal{U}(s) &= \frac{1}{2}\left(-16s^2\mathrm{Y}_1'(s)-16s\mathrm{Y}_1'(s)-16\mathrm{Y}_1(s)-16s\mathrm{Y}_2'(s)-16\mathrm{Y}_2'(s)+16\mathrm{Y}_2(s)+s-2\log(s+3)+1\right)\end{aligned} \quad (9.30)$$

On the other hand we find that $\mathcal{A} = 0$ while the $\mathcal{B}$-term factorises as follows:

$$\begin{aligned} \mathcal{B} &= \mathcal{H}(t)J(s) & (9.31)\\ \mathcal{H}(t) &= \frac{4\Pi''(t)}{t+3} & (9.32)\\ J(s) &= \frac{16s(s+3)\mathrm{Y}_1''(s)+32(s+3)\mathrm{Y}_1'(s)+16s\mathrm{Y}_2''(s)+48\mathrm{Y}_2''(s)-1}{s+3} & (9.33)\end{aligned}$$

In this way the solution of the MA equation reduces to the solution of two separate integral differential equations one in the s variable, one in the t-variable:

$$\frac{\lambda}{2}\,t\,\exp[T(t)] \;=\; \mathcal{H}(t) \quad ; \quad \frac{\mu}{2}\,s\,\exp[\mathcal{U}(s)] \;=\; J(s) \quad (9.34)$$

We focus on the first in the variable t. With rather simple manipulations it can be reduced to an ordinary differential equation of higher order, namely:

$$\frac{8(t+1)\Pi''(t)}{t+3} - \frac{\Pi^{(3)}(t)}{\Pi''(t)} + \frac{2t+3}{t^2+3t} \;=\; 0 \quad (9.35)$$

which is a differential equation of the first order for the function $F(t)$. Apart from an integration constant which is fixed by the topological constraints on the periods of Ricci form, the unique solution of eq. (9.35) is $F(t,0)$ corresponding to the geometry of $\mathbb{W}P[112]$. This shows that in order to impose a boundary function consistent with $\alpha \neq 0$ we need to modify the ansatz (9.27) in such a way as to introduce a certain s,t-mixing.

10 Conclusions

As we advocated in the introduction, the present paper is an illustration of the conjecture 1.1 for which we have strong support from the fact that it is verified for the value $\Delta = 0$ of the parameter in the paradigmatic case of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity resolution. Further numerical evidence emerges from the study of the power series solution of the Monge-Ampère equation in the symplectic potential formulation. This latter in its hybrid version seems to provide the most promising approach since different series expansions might be glued together to prolong the solution beyond the valleys of oscillations.

Assuming that in due time our conjecture can be transformed into a proof, we would like to stress its relevance. According to our view point, Conjecture 1.1 provides a precise mathematical relationship to realise the gauge/gravity correspondence in a proper way. The generalized Kronheimer construction fixes all the items of the gauge theory on the brane world-volume: field content, gauge group, flavor symmetries and interactions. As maintained by Conjecture 1.1, the same Kronheimer construction determines, via the Monge-Ampère equation, also the Ricci-flat Kähler metric to be used in the construction of the dual D3-brane solution of supergravity. If 1.1 is proved we can say that, for the class of theories realised on D3 branes at $\mathbb{C}^3/\Gamma$ Calabi-Yau singularities, the McKay quiver determines uniquely both sides of the correspondence.

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A The affine variety $\mathbb{C}^3/\mathbb{Z}_4$

We study the quotient $\mathbb{C}^3/\mathbb{Z}_4$ as an affine variety, i.e., as a closed (in the Zariski topology) subset of an affine space $\mathbb{C}^D$ cut by an ideal I of the polynomial ring $\mathbb{C}[x_1, \ldots, x_D]$. In particular, we show that $\mathbb{C}^3/\mathbb{Z}_4$ is not a (schematic) complete intersection.

Reminder on complete intersections. We recall the notions of *set-theoretic* and *schematic* intersection.

Definition A.1. *Let $X \subset \mathbb{C}^D$ be an affine variety, and denote by d its codimension in $\mathbb{C}^D$.*

1. *X is a set-theoretic complete intersection if it is cut by d equations as a subset of $\mathbb{C}^D$.*
2. *X is a schematic complete intersection if it is cut by d equations as an affine variety. In other terms, if $A(X)$ is the coordinate ring of X (the ring of regular functions on X), then $A(X) = \mathbb{C}[x_1, \ldots, x_D]/I$, where the minimal number of generators of the ideal I is d.*

It turns out that all quotients $\mathbb{C}^3/\Gamma$, where Γ is a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$, are set-theoretic complete intersections, and therefore so is the case for $\mathbb{C}^3/\mathbb{Z}_4$. However, we shall not prove this fact here, and rather concentrate on proving that $\mathbb{C}^3/\mathbb{Z}_4$ is not a schematic complete intersection.

Finding the equations. An affine toric variety X can be expressed as

$$X_\sigma = \mathrm{Specm}\ \mathbb{C}[S_\sigma] \tag{A.1}$$

where $\sigma \subset N \otimes \widehat{R}$ is a strongly convex polyhedral cone, N is a lattice, and S_σ is the semigroup $S_\sigma = \sigma^\vee \cap M$, with $\sigma^\vee$ the dual cone to σ; M is the dual of the lattice N. Specm denotes the maximal spectrum, i.e., the set of maximal ideals of $\mathbb{C}[S_\sigma]$ with the Zariski topology. Basically following [64], we delineate a procedure to find the equations for the affine toric variety X. We remind that a *Hilbert basis* $\mathcal{H}_\sigma$ for the semigroup S_σ is a minimal set of generators for S_σ which contains the rational generators of the rays of $\sigma^\vee$. Define $D = \#\mathcal{H}_\sigma$. Then the elements of $\mathcal{H}_\sigma$ are related by $D-n$ relations, which generate an ideal $I_{\sigma,0}$ of $\mathbb{C}[x_1,\ldots,x_D]$. Given two ideals I, J in a ring R, the saturation of I with respect to J is defined as

$$I : J^\infty = \{a \in R \,|\, a^N J \subset I \text{ for } N \gg 0.\}$$

Then one proves that the ideal I_σ of X_σ in $\mathbb{C}^D$ is the saturation of $I_{\sigma,0}$ wih the respect to the ideal

$$K_\sigma = (x_1 \cdots x_D) \subset \mathbb{C}[x_1,\ldots,x_D].$$

Remark A.1. *If I, J are ideals in $\mathbb{C}[x_1,\ldots,x_D]$, the affine variety corresponding to the ideal $I : J^\infty$ is*

$$V(I : J^\infty) = \overline{V(I) \setminus V(J)}$$

where the closure is taken in $\mathbb{C}^D$ (in the Zariski topology), and for every ideal L in $\mathbb{C}[x_1,\ldots,x_D]$, $V(L)$ denotes the closed set in $\mathbb{C}^D$ corresponding to L.

Equations for $X = \mathbb{C}^3/\mathbb{Z}_4$. Now we check that $\mathbb{C}^3/\mathbb{Z}_4$ is *not* a schematic complete intersection, as noted in [65]. Realizing X as in equation (A.1) we can take for σ the cone with generators $(1,0,0)$, $(-1,2,0)$, $(0,-1,2)$ in the latttice $N = \mathbb{Z}^3$. The dual cone $\sigma^\vee$ has rational generators $(4,2,1)$, $(0,2,1)$, $(0,0,1)$ in $M \simeq \mathbb{Z}^3$. A Hilbert basis of S_σ is obtained by adding the lattice points

$$(1,1,1),\ (0,1,1),\ (1,2,1),\ (2,1,1),\ (2,2,1),\ (3,2,1).$$

Assigning variables $x_1,\ldots,x_9$ to these lattice points we obtain that $I_{\sigma,0}$ is generated by the 6 equations

$$\begin{gathered} x_1x_8 - x_9^2 = 0, \quad x_2x_9^2 - x_8^3 = 0, \quad x_3x_9^2 - x_7^2x_8 = 0, \\ x_4x_9 - x_7x_8 = 0, \quad x_5x_9^2 - x_7x_8^2 = 0, \quad x_6x_9 - x_8^2 = 0 \end{gathered}$$

Saturating this ideal with respect to $K = (x_1 \cdots x_9)$ one sees that I_σ is generated by the 20 quadratic equations (the equation needed to cut X from $\mathbb{C}^9$ with the correct schematic structure):

$$\begin{array}{ccccc} x_8^2 - x_6x_9; & x_7x_8 - x_4x_9; & x_6x_8 - x_2x_9; & x_4x_8 - x_4x_5; & x_1x_8 - x_9^2; \\ x_6x_7 - x_5x_9; & x_5x_7 - x_3x_8; & x_4x_7 - x_3x_9; & x_2x_7 - x_5x_8; & x_6^2 - x_2x_8; \\ x_4x_6 - x_5x_8; & x_1x_6 - x_8x_9; & x_4x_5 - x_3x_6; & x_1x_5 - x_4x_9; & x_4^2 - x_3x_8; \\ x_2x_4 - x_5x_6; & x_1 - x_7x_9; & x_2x_3 - x_5^2; & x_1x_3 - x_7^2; & x_1x_2 - x_6x_9. \end{array} \tag{A.2}$$

These are a *minimal set* of generators. So X is the intersection of 20 quadrics in $\mathbb{C}^9$. All these quadrics are singular along their intersection with a plane of codimension 3 (when their equation contains a square) or 4 (when their equation does not contain a square). The dimension of the singular locus is 6 and 5 respectively (not 5 and 4!) It may be interesting to see what variety does the ideal $I_{\sigma,0}$ describe. To this end one computes the primary decomposition of the ideal [66]. This yields 5 ideals; one is radical, and coincides with I_σ, so that one component of the variety is $\mathbb{C}^3/\mathbb{Z}_4$. The other ideals are generated by monomials, and correspond to (intersections of) coordinate planes of different dimensions, counted with multiplicities.

B The orbifold $\mathbb{S}^5/\mathbb{Z}_4$

Setting $s = -\frac{2}{3}R^2$ with $r \to \infty$ in the metric (5.9), it is straightforward to verify that this takes the approximate form

$$\mathrm{ds}^2_{\mathrm{tot}K_{\mathbb{W}P[112]}} \overset{R\to\infty}{\approx} dR^2 + R^2 \mathrm{ds}^2_{X_5} \tag{B.1}$$

at leading order in R. Since the metric is Ricci-flat Kähler, and it takes the form of a cone over a five-dimensional space, it follows that locally the five-dimensional metric $\mathrm{ds}^2_{X_5}$ is Sasaki-Einstein. Below we shall show that globally, this is precisely a Sasaki-Einstein metric on the orbifold $\mathbb{S}^5/\mathbb{Z}_4$.

In the coordinates used in the paper, the five-dimensional metric reads

$$\begin{aligned}\mathrm{ds}^2_{X_5} \;=\; & -\frac{t}{6}\left(\sin^2\theta d\phi^2 + d\theta^2\right) - \frac{dt^2}{2t(2t+3)} - \frac{2t(2t+3)}{9}\left[\frac{d\chi}{3} - \frac{1}{2}[(1-\cos\theta)d\phi + d\tau]\right]^2 \\ & + \frac{4}{9}\left[\left(\frac{t}{3}+1\right)d\chi - \frac{1}{2}t[(1-\cos\theta)d\phi + d\tau]\right]^2\end{aligned} \tag{B.2}$$

After introducing the new coordinate $\sigma \in [0, \frac{\pi}{2}]$ as

$$t = -\frac{3}{2}\sin^2\sigma \tag{B.3}$$

it becomes

$$\begin{aligned}\mathrm{ds}^2_{X_5} \;=\; & d\sigma^2 + \frac{\sin^2\sigma}{4}\left(\sin^2\theta d\phi^2 + d\theta^2\right) + \frac{\sin^2\sigma\cos^2\sigma}{4}\left[\frac{2}{3}d\chi - d\tau - d\phi + \cos\theta d\phi\right]^2 \\ & + \frac{1}{9}\left[2d\chi - \frac{3}{2}\sin^2\sigma\left(\frac{2}{3}d\chi - d\tau - d\phi + \cos\theta d\phi\right)\right]^2\end{aligned} \tag{B.4}$$

and one can check that this is indeed locally a Sasaki-Einstein metric, where the first line is a Kähler-Einstein metric. In order to uncover the relation with the metric on the five-sphere $\mathbb{S}^5$, it is convenient to redefine the angular coordinates as

$$\widetilde{\phi} = \phi\,, \qquad \beta = \tfrac{2}{3}\chi - \phi - \tau\,, \qquad \psi = 2\chi\,, \tag{B.5}$$

with inverse

$$\phi = \widetilde{\phi}\,, \qquad \tau = \tfrac{1}{3}\psi - \phi - \beta\,, \qquad \chi = \tfrac{1}{2}\psi\,, \tag{B.6}$$

where, after performing the change of coordinates, we can drop the tilde on $\widetilde{\phi}$ and simply continue to denote this as ϕ. The metric then reads

$$\begin{aligned}\mathrm{ds}^2_{X_5} \;=\; & d\sigma^2 + \frac{\sin^2\sigma}{4}\left(\sin^2\theta d\phi^2 + d\theta^2\right) + \frac{\sin^2\sigma\cos^2\sigma}{4}\left(d\beta + \cos\theta d\phi\right)^2 \\ & + \frac{1}{9}\left[d\psi - \frac{3}{2}\sin^2\sigma\left(d\beta + \cos\theta d\phi\right)\right]^2\end{aligned} \tag{B.7}$$

It is well-known (and simple to verify) that taking $\phi \sim \phi + 2\pi$ and $\beta \sim \beta + 4\pi$, with $\theta \in [0,\pi]$ and $\sigma \in [0, \frac{\pi}{2}]$, the first line is the standard Einstein metric on $\mathbb{P}^2$. Moreover, with $\psi \sim \psi + 6\pi$, the five-dimensional metric is the round metric on $\mathbb{S}^5$, viewed as the total space of a circle bundle $\mathbb{S}^5 \xrightarrow{\pi} \mathbb{P}^2$, normalised so to obey the equation

$$R^{X_5}_{ij} = 4 g^{X_5}_{ij} \tag{B.8}$$

On the other hand, we are not free to chose the ranges of the coordinates, but these are inherited from the ranges of the original coordinates (ϕ, τ, χ), fixed in (5.1). From the change of coordinates (B.5), it follows[23] that we must enforce the following periodicities:

$$\phi \sim \phi + 2\pi \,, \qquad \beta \sim \beta + 2\pi \,, \qquad \psi \sim \psi + 3\pi \ , \tag{B.9}$$

thus suggesting that globally the space is an orbifold $\mathbb{S}^5/\mathbb{Z}_4$. However, the precise form of the $\mathbb{Z}_4$ action is not transparent from these considerations.

Next, we will show that the $\mathbb{Z}_4$ action is precisely the correct action inherited from the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold singularity. We start with three standard complex coordinates (z_1, z_2, z_3) on $\mathbb{C}^3$ and consider the following change of coordinates

$$z_1 = R \sin\sigma \cos\tfrac{\theta}{2} e^{\mathrm{i}\left(-\frac{\beta+\phi}{2}+\frac{\psi}{3}\right)} \,, \qquad z_2 = R \sin\sigma \sin\tfrac{\theta}{2} e^{\mathrm{i}\left(\frac{\phi-\beta}{2}+\frac{\psi}{3}\right)} \,, \qquad z_3 = R\cos\sigma e^{\mathrm{i}\frac{\psi}{3}} \ , \tag{B.10}$$

where

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = R^2 \tag{B.11}$$

It can be checked that the metric induced at $R = 1$,

$$\mathrm{ds}^2_5 = (|dz_1|^2 + |dz_2|^2 + |dz_3|^2)|_{R=1} \tag{B.12}$$

coincides with (B.7), and more generally, the six-dimensional metric is the cone $\mathrm{ds}^2_{\text{cone}} = dR^2 + R^2 \mathrm{ds}^2_{X_5}$.

To see that these are good coordinates on $\mathbb{C}^3$ we can also view it as $\mathbb{C}^3 \simeq \mathbb{R}^6 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, by defining

$$z_1 = \rho_1 e^{\mathrm{i}\varphi_1} \,, \qquad z_2 = \rho_2 e^{\mathrm{i}\varphi_2} \,, \qquad z_3 = \rho_3 e^{\mathrm{i}\varphi_3} \tag{B.13}$$

so that the induced metric reads

$$\mathrm{ds}^2_{\text{cone}} = |dz_1|^2 + |dz_2|^2 + |dz_3|^2 = d\rho_1^2 + \rho_1^2 d\varphi_1^2 + d\rho_2^2 + \rho_2^2 d\varphi_2^2 + d\rho_3^2 + \rho_3^2 d\varphi_3^2 \tag{B.14}$$

where for $\mathbb{C}^3$ the ranges of the coordinates are now $\rho_i \in [0, +\infty)$ and $\varphi_i \sim \varphi_i + 2\pi$, for $i = 1, 2, 3$. Defining $y_i = \frac{1}{2}\rho_i^2$, this gives the standard metric in symplectic-toric coordinates, with Kähler form

$$\mathbb{K}_{\mathbb{C}^3} = dy_1 \wedge d\varphi_1 + dy_2 \wedge d\varphi_2 + dy_3 \wedge d\varphi_3 \tag{B.15}$$

From this it is clear that the $U(1)^3$ torus action on $\mathbb{C}^3$

$$(z_1, z_2, z_3) \to (\lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3) \tag{B.16}$$

with $|\lambda_i| = 1$, $\lambda_i = e^{\mathrm{i}c_i}$ descends on the φ_i coordinates to

$$(\varphi_1, \varphi_2, \varphi_3) \to (\varphi_1 + c_1, \varphi_2 + c_2, \varphi_3 + c_3) \tag{B.17}$$

Notice that on $\mathbb{C}^3$ the periodicities of the two sets of angular coordinates are consistent with the change of coordinates

$$\varphi_1 = -\frac{\beta+\phi}{2} + \frac{\psi}{3} \,, \qquad \varphi_2 = \frac{-\beta+\phi}{2} + \frac{\psi}{3} \ , \qquad \varphi_3 = \frac{\psi}{3} \,, \tag{B.18}$$

[23]The periodicities of ϕ and ψ are obvious. The simplest way to determine the periodicity of β is by demanding that the total volume of the three-torus with coordinates (ϕ, χ, τ) is preserved by the coordinate transformation (B.5).

with inverse

$$\phi = -\varphi_1 + \varphi_2, \qquad \beta = -\varphi_1 - \varphi_2 + 2\varphi_3 \, , \qquad \psi = 3\varphi_3 \, , \tag{B.19}$$

as we have

$$(2\pi)^3 = \int d\varphi_1 d\varphi_2 d\varphi_3 = \frac{1}{6}\int d\phi d\beta d\psi = \frac{1}{6}(2\pi)(4\pi)(6\pi) \tag{B.20}$$

Let us now reformulate the standard orbifold action of a discrete group $\Gamma \in SU(3)$ on $\mathbb{C}^3$ with the corresponding action on $\mathbb{S}^5$ in the above (ϕ, β, ψ) coordinates. We will restrict to $\Gamma = \mathbb{Z}_n$ for simplicity. In the (z_1, z_2, z_3) coordinates on $\mathbb{C}^3$, a $\mathbb{Z}_n$ orbifold action is defined by the identification

$$(z_1, z_2, z_3) \sim (\omega_n^{a_1} z_1, \omega_n^{a_2} z_2, \omega_n^{a_3} z_3) \tag{B.21}$$

where ω_n is a n-th root of unity. The requirement that $\mathbb{Z}_n \in SU(3)$ implies that

$$a_1 + a_2 + a_3 = 0 \mod n \tag{B.22}$$

Using (B.17), the above orbifold action implies the following identification in the φ_i coordinates

$$(\varphi_1, \varphi_2, \varphi_3) \sim (\varphi_1 + a_1 \tfrac{2\pi}{n}, \varphi_2 + a_2 \tfrac{2\pi}{n}, \varphi_3 + a_3 \tfrac{2\pi}{n}) \tag{B.23}$$

and, equivalently, the following identification in the (ϕ, β, ψ) coordinates

$$(\phi, \beta, \psi) \sim (\phi + (-a_1 + a_2)\tfrac{2\pi}{n}, \beta + (-a_1 - a_2 + 2a_3)\tfrac{2\pi}{n}, \psi + 3a_3 \tfrac{2\pi}{n}) \tag{B.24}$$

The simplest example is the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, with $\mathbb{Z}_3$ action on $\mathbb{C}^3$ given by

$$(z_1, z_2, z_3) \sim (e^{\mathrm{i}\frac{2\pi}{3}} z_1, e^{\mathrm{i}\frac{2\pi}{3}} z_2, e^{\mathrm{i}\frac{2\pi}{3}} z_3),$$

which using (B.24) corresponds simply to $\psi \sim \psi + 2\pi$. In this case, the metric (B.7), taking $\phi \in [0, 2\pi]$, $\beta \in [0, 4\pi]$, is the metric on $\mathbb{S}^5/\mathbb{Z}_3$. This space can also be viewed as the unit circle bundle inside $\mathcal{O}_{\mathbb{P}^2}(-3)$, namely the total space of the canonical line bundle over $\mathbb{P}^2$.

Let us now discuss our main example, the orbifold $\mathbb{C}^3/\mathbb{Z}_4$. In the table below we summarise the action of the three non-trivial elements of $\mathfrak{g} \in \mathbb{Z}_4$, including the identifications both in the $(\varphi_1, \varphi_2, \varphi_3)$ and the (ϕ, β, ψ) coordinates.

$\mathfrak{g} : (z_1, z_2, z_3)$	$\{a_1, a_2, a_3\}$	$(\varphi_1, \varphi_2, \varphi_3) \sim$	$(\phi, \beta, \psi) \sim$
$(\mathrm{i}, \mathrm{i}, -1)$	$\{1, 1, 2\}$	$(\varphi_1 + \frac{\pi}{2}, \varphi_2 + \frac{\pi}{2}, \varphi_3 + \pi)$	$(\phi, \beta + \pi, \psi + 3\pi)$
$(-1, -1, 1)$	$\{2, 2, 0\}$	$(\varphi_1 + \pi, \varphi_2 + \pi, \varphi_3)$	$(\phi, \beta + 2\pi, \psi)$
$(-\mathrm{i}, -\mathrm{i}, -1)$	$\{3, 3, 2\}$	$(\varphi_1 + \frac{3\pi}{4}, \varphi_2 + \frac{3\pi}{4}, \varphi_3 + \pi)$	$(\phi, \beta + 3\pi, \psi + 3\pi)$

(B.25)

As we see, in either of these two sets of angular coordinates the identifications are not diagonal. In the coordinates (ϕ, β, ψ) the clearest identification is the action of (junior) element $\{2, 2, 0\}$, which implies that the base space, with metric in the first line of (B.7), is $\mathbb{P}^2/\mathbb{Z}_2$. The action of the (junior) element $\{1, 1, 2\}$ means that as β goes half way around its circle, the coordinate ψ goes once around the ψ-circle, with period 3π. The action of the (senior) element $\{3, 3, 2\}$ is simply a consequence of the previous two.

In order to clarify the orbifold action on S^5, it is useful to adopt a set of angular coordinates in which the $\mathbb{Z}_4$ action is diagonal. It is then simple to verify that this is achieved precisely by the original coordinates (ϕ, τ, χ) defined in (5.1). We summarise this diagonal action in the table below, where for convenience we defined $\gamma \equiv \frac{4}{3}\chi$.

$\mathfrak{g} : (z_1, z_2, z_3)$	$\{a_1, a_2, a_3\}$	$(\phi, \tau, \gamma) \sim$
$(\mathrm{i}, \mathrm{i}, -1)$	$\{1, 1, 2\}$	$(\phi, \tau, \gamma + 2\pi)$
$(-1, -1, 1)$	$\{2, 2, 0\}$	$(\phi, \tau + 2\pi, \gamma)$
$(-\mathrm{i}, -\mathrm{i}, -1)$	$\{3, 3, 2\}$	$(\phi, \tau + 2\pi, \gamma + 2\pi)$

(B.26)

This shows that the indeed, the $\mathbb{Z}_4$ action on $\mathbb{C}^3$ induces the correct $\mathbb{Z}_4$ action on the asymptotic metric on $\mathbb{S}^5$.

In order to further clarify the orbifold action on $\mathbb{S}^5$, it is convenient to rewrite the metric (B.7) in the form of a circle fibration over a base space, that turns out to be precisely $\mathbb{W}P[112]$. In particular, rearranging the terms in (B.7) we find

$$\mathrm{ds}^2_{X_5} = \widetilde{\mathrm{ds}}^2_{\mathbb{W}P[112]} + \frac{1}{16}(1 + 3\cos^2\sigma)\left[d\gamma + \frac{2\sin^2\sigma}{1 + 3\cos^2\sigma}(d\tau - \cos\theta d\phi)\right]^2 \tag{B.27}$$

where[24]

$$\widetilde{\mathrm{ds}}^2_{\mathbb{W}P[112]} = d\sigma^2 + \frac{1}{4}\sin^2\sigma\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{\sin^2\sigma\cos^2\sigma}{1 + 3\cos^2\sigma}\left(d\tau - \cos\theta d\phi\right)^2 , \tag{B.28}$$

which clearly displays the fact that $S^5/\mathbb{Z}_4$ arises as the total space of a circle fibration over $\mathbb{W}P[112]$, equipped with the metric (B.28). We decorated this metric with a tilde to distinguish it from the different metric on $\mathbb{W}P[112]$, that we discuss in the main body of the paper, namely the metric (8.2) induced on the exceptional divisor by the Ricci-flat metric (5.9). Below we will rewrite the latter metric in different coordinates, to facilitate the comparison with the metric in (B.28). Let us discuss briefly how to see that the underlying (singular) variety to the metric defined in (B.28) is indeed $\mathbb{W}P[112]$. With the ranges of coordinates and periodicities $\sigma \in [0, \frac{\pi}{2}]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, $\tau \in [0, 2\pi]$ we see that near to $\sigma \approx 0$ the metric develops an $\mathbb{R}^4/\mathbb{Z}_2$ singularity (it is a cone over the Lens space $\mathbb{S}^3/\mathbb{Z}_2$), while near to $\sigma \approx \frac{\pi}{2}$, the space shrinks smoothly to $\mathbb{S}^2 \times \mathbb{R}^2$. Following a reasoning analogous to that in the main body of the paper, one can see that there exists only one non-trivial two-cycle

$$C_2 \quad \Leftrightarrow \quad \{\theta = \text{constant},\ \phi = \text{constant}\} \tag{B.29}$$

while the other two-cycle of $\mathbb{F}_2$, that would be defined by $C_1 \Leftrightarrow \{t = t_{max} = -\frac{3}{2}\sin^2\sigma_{max} \neq 0\}$ is shrunk to zero size in the above metric[25].

From the metric (B.27) we now read off the connection one-form

$$\widetilde{\mathcal{A}} \equiv \frac{2\sin^2\sigma}{1 + 3\cos^2\sigma}(d\tau - \cos\theta d\phi) \tag{B.30}$$

whose associated first Chern class can be integrated on C_2 to give

$$\frac{1}{2\pi}\int_{C_2} d\widetilde{\mathcal{A}} = 2 \tag{B.31}$$

showing that indeed this is a connection on the unit circle bundle inside the canonical bundle of $\mathbb{W}P[112]$.

[24]Interestingly, precisely this metric was found in [67] as a limiting case of a more general one-parameter family of smooth metrics on $\mathbb{F}_2$, in the context of AdS_5 solutions of eleven-dimensional supergravity. See eq. (5.7) of this reference.

[25]Since we have not established whether the above metric is Kähler, the simplest way to see this is probably to consider the explicit one-parameter familiy of metrics on $\mathbb{F}_2$ desingularising (B.28), presented in [67].

We conclude this appendix by writing the metric on the exceptional divisor (8.2) induced by the Ricci-flat metric (5.9) in a form that makes more transparent the comparison with the above discussion. Using (B.3) we have that

$$\mathrm{ds}^2_{\mathcal{ED}} = \frac{9}{4}\left[(1+\cos^2\sigma)d\sigma^2 + \frac{1}{2}\sin^2\sigma\left(d\theta^2+\sin^2\theta d\phi^2\right) + \frac{\sin^2\sigma\cos^2\sigma}{1+\cos^2\sigma}\left(d\tau-\cos\theta d\phi\right)^2\right] \tag{B.32}$$

from which the similarity with the metric (B.28) is apparent. For completeness, let us also display the behaviour of the orthotoric metric (5.9) near to the exceptional divisor. Setting $s=-3-\rho^2$ in (5.9), for $\rho\to 0$ we have

$$\mathrm{ds}^2_{\mathrm{tot}K_{\mathbb{W}P[112]}} \overset{\rho\to 0}{\approx} \mathrm{ds}^2_{\mathcal{ED}} + \frac{3(1+\cos^2\sigma)}{8}\left[d\rho^2+\rho^2\left[d\gamma+\frac{2\sin^2\sigma}{1+\cos^2\sigma}(d\tau-\cos\theta d\phi)\right]^2\right] \tag{B.33}$$

where the angular variables $(\phi,\tau,\gamma\equiv\frac{4}{3}\chi)$ are precisely those defined in (5.1), which all have canonical 2π-periodicities. This shows that the metric (5.9) is smooth in the neighborhood of the exceptional divisor $\mathcal{ED}=\mathbb{W}P[112]$, in particular locally it has the topology of $\mathbb{W}P[112]\times\mathbb{C}^2$. The connection one-form

$$\mathcal{A}\equiv\frac{2\sin^2\sigma}{1+\cos^2\sigma}(d\tau-\cos\theta d\phi) \tag{B.34}$$

read off from (B.33) has first Chern class again given by

$$\frac{1}{2\pi}\int_{C_2} d\mathcal{A} = 2 \tag{B.35}$$

as it should be.

To summarise, in this appendix we have shown that the orbifold action of $\mathbb{Z}_4$ on $\mathbb{S}^5$, induced by the $\mathbb{C}^3/\mathbb{Z}_4$ quotient, is not diagonal in the canonical coordinates where the Sasaki-Einstein metric on $\mathbb{S}^5$ can be viewed as a $U(1)$ fibration over $\mathbb{P}^2$ with its Kähler-Einstein metric. This action is *diagonalised* precisely by the coordinates $(\phi,\tau,\frac{4}{3}\chi)$ used in the main part of the paper, and adapting the metric to these coordinates, it takes the form of a $U(1)$ fibration over $\mathbb{W}P[112]$, with the non-Einstein metric (B.28). This is precisely the unit circle bundle in the canonical line bundle over $\mathbb{W}P[112]$. The metric on the exceptional divisor of the partial resolution, induced from the orthotoric Ricci-flat metric, is a similar, but manifestly *different* non-Einstein metric (B.32).

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